NOAA Technical Report NOS 61

Radiation Pressure on a Spheroidal Satellite

JAMES R. LUCAS

ROCKVILLE, MD.
JULY 1974

noaa NATIONAL OCEANIC AND ATMOSPHERIC ADMINISTRATION / NATIONAL OCEAN SURVEY

NOAA TECHNICAL REPORTS

National Ocean Survey Series

The National Ocean Survey (NOS) provides charts and related information for the safe navigation of marine and air commerce. The survey also furnishes other earth science data--from geodetic, hydrographic, oceanographic, geomagnetic, seismologic, gravimetric, and astronomic surveys, observations, investigations, and measurements--to protect life and property and to meet the needs of engineering, scientific, defense, commercial, and industrial interests.

Because many of these reports deal with new practices and techniques, the views expressed are those of the authors and do not necessarily represent final survey policy. NOS series NOAA Technical Reports is a continuation of, and retains the consecutive numbering sequence of, the former series, Environmental Science Services Administration (ESSA) Technical Reports Coast and Geodetic Survey (C&GS), and the earlier series, C&GS Technical Bulletins.

Those publications marked by an asterisk are out of print. The others are available through the Superintendent of Documents, U.S. Government Printing Office, Washington, D.C. 20402. Price as indicated. Beginning with 39, microfiche is available at the National Technical Information Service (NTIS), U.S. Department of Commerce, Sills Bldg., 5285 Port Royal Road, Springfield, Va. 22151. Price $1.45. Order by accession number, when given, in parentheses.

COAST AND GEODETIC SURVEY TECHNICAL BULLETINS

*No. 22 Tidal Current Surveys by Photogrammetric Methods. Morton Keller, October 1963.

*No. 23 Aerotriangulation Strip Adjustment. M. Keller and G. C. Tewinkel, August 1964.

*No. 24 Satellite Triangulation in the Coast and Geodetic Survey. February 1965.

*No. 25 Aerotriangulation: Image Coordinate Refinement. M. Keller and G. C. Tewinkel, March 1965.

*No. 26 Instrumented Telemetering Deep Sea Buoys. H. W. Straub, J. M. Arthaber, A. L. Copeland, and D. T. Theodore, June 1965.

*No. 27 Survey of the Boundary Between Arizona and California. Lansing G. Simmons, August 1965.

*No. 28 Marine Geology of the Northeastern Gulf of Maine. R. J. Malloy and R. N. Harbison, February 1966.

*No. 29 Three-Photo Aerotriangulation. M. Keller and G. C. Tewinkel, February 1966.

*No. 30 Cable Length Determinations for Deep-Sea Oceanographic Operations. Robert C. Darling, June 1966.

*No. 31 The Automatic Standard Magnetic Observatory. L. R. Alldredge and I. Saldukas, June 1966.

ESSA TECHNICAL REPORTS

*C&GS 32 Space Resection in Photogrammetry. M. Keller and G. C. Tewinkel, September 1966.

*C&GS 33 The Tsunami of March 28, 1964, as Recorded at Tide Stations. M. G. Spaeth and S. C. Berkman, July 1967.

*C&GS 34 Aerotriangulation: Transformation of Surveying and Mapping Coordinate Systems. Melvin J. Umbach, August 1967.

*C&GS 35 Block Analytic Aerotriangulation. M. Keller and G. C. Tewinkel, November 1967.

*C&GS 36 Geodetic and Grid Angles--State Coordinate Systems. Lansing G. Simmons, January 1968.

*C&GS 37 Precise Echo Sounding in Deep Water. George A. Maul, January 1969.

*C&GS 38 Grid Values of Total Magnetic Intensity IGRF--1965. E. B. Fabiano and N. W. Peddie, April 1969.

C&GS 39 An Advantageous, Alternative Parameterization of Rotations for Analytical Photogrammetry. Allen J. Pope, September 1970. Price $0.30 (COM-71-00077)

C&GS 40 A Comparison of Methods for Computing Gravitational Potential Derivatives. L. J. Gulick, September 1970. Price $0.40 (COM-71-00185)

(Continued on inside back cover)

CONTENTS

For sale by the Superintendent of Documents, U.S. Government Printing Office
Washington, D.C. 20402 - Price 75 cents
Stock Number 0321-00009

RADIATION PRESSURE ON A SPHEROIDAL SATELLITE

James R. Lucas
Geodetic Research and Development Laboratory
National Ocean Survey, NOAA
Rockville, Maryland

ABSTRACT. The pressure of solar radiation on a spherical balloon satellite is proportional to its cross-sectional area. However, there is evidence to indicate that the PAGEOS satellite has not remained spherical, but is more nearly a prolate spheroid that is rotating about its minor axis. If this is true, the force of solar radiation incident upon its surface must be expressed in terms of the shape of the surface and its orientation to the sun. Furthermore, radiation reflected from the surface of an aspherical balloon will impart an additional force which can be large enough to significantly perturb the orbit. By starting with basic equations for the radiation forces on a flat plate and integrating over the sunlit portion of the surface, exact expressions are obtained for both the incident and reflected radiation forces on a stationary, prolate spheroidal satellite. These expressions are then used to develop power series expansions for the radiation forces on a rotating spheroid.

INTRODUCTION

The perturbing effects of solar radiation pressure on the motions of large balloon satellites have been known for a number of years. Perturbation formulas for spherical satellites have been published by several authors (Musen 1960, Bryant 1961, Kozai 1963). To the first order these formulas have been quite satisfactory in explaining the motions of satellites such as ECHO I. However, studies by Fea and Smith (1970) show that both 1963-30D (DASH 2) and 1966-56A (PAGEOS) have undergone accelerations, apparently related to radiation pressure, which cannot be explained by the first-order theory.

Photoelectric photometry studies of PAGEOS conducted by Vanderburgh and Kissell (1970) indicate that PAGEOS is no longer spherical, but has gradually deformed until its shape is more nearly approximated by a prolate spheroid. In addition the spheroid appears to be rotating about an axis which is precessing about the satellite–sun vector. This information led Smith and Fea (1970) to develop a modification to the classical theory, which takes into account both the varying cross–section presented to the incident radiation by a rotating, aspherical balloon and the force imparted to the satellite when radiation is reflected from its surface. Smith and Kissell (1971) showed that this modified theory provides a much better fit to the observed periodic fluctuations in the mean motion of PAGEOS during a period (July through October 1969) when the orbit was partly in the earth's shadow.

A computer program based on this modified theory was developed at the Geodetic Research and Development Laboratory, and tests with this program (Lucas and Chovitz 1972) verified, in part, the results of Smith and Kissell. However, the discrepancies between observed orbital positions and those predicted by the program were still excessively large. Upon re–examination of Smith and Fea's work it appeared that they had perhaps oversimplified their mathematical model by making two major assumptions: (1) That the effective direction of radiation reflected specularly from the balloon's surface is determined by Snell's law applied to the incident ray that passes through the center of the satellite, and (2) that the magnitude of the flux reflected in this direction approximates that which would be reflected by a sphere of surface area equal to that of the spheroid. While these assumptions appear reasonable, the results of rough calculations were to the contrary.

Since the modified theory of Smith and Fea was believed to be sound, it was decided to attempt a more rigorous mathematical formulation of the problem. In this report exact expressions are developed for the radiation forces acting on a stationary prolate spheroid and one that is rotating about its major axis. Force equations are also developed for a spheroid rotating about a minor axis, but in this case exact expressions are not possible, so series approximations are employed.

DEVELOPMENT OF THE BASIC FORCE EQUATIONS

Consider a flat plate located in space above the earth's atmosphere and oriented normal to the sun's rays. The plate will be constantly bombarded by a stream of photons from the sun. The intensity of this solar power, which will be denoted by I, varies inversely as the distance to the sun's center. Its value at one astronomical unit, called the solar constant and denoted by I_0, is reported by Thekaekara and Drummond (1971) to be about 0.135 watts/cm^2.

Since force is power divided by velocity, the force per unit area exerted on the plate by incident solar radiation is I/c, where c is the velocity of light. The direction of this force is obviously the same as the direction of the incident radiation.

Of the radiation striking the plate, some fraction, R, will be reflected and the remainder will be absorbed and reradiated. Of the reflected radiation some fraction, R_S, will be reflected specularly and the remainder, R_D, will be reflected diffusely. By Newton's third law the force imparted to the plate by specularly reflected radiation will be equal in magnitude to the incident force multiplied by R_S and will be directed opposite to the direction taken by the reflected radiation.

If we assume that the diffusely reflected radiation obeys Lambert's reflection law, the resultant force due to reflection of this type will have magnitude equal to the incident force multiplied by $2R_D/3$ (Georgevic 1971). This force will be directed away from the sun and act along the normal to the plate.

Although it would not be difficult to treat diffusely reflected radiation force by the procedure to be developed in the following sections, this report is directed toward balloon satellites that have been coated with materials of high specular reflectivity. Therefore, throughout the remainder of this report we will assume that, for the surfaces considered, the specular albedo, R_S, is very much larger than the diffuse albedo, R_D. Forces arising from diffusely reflected radiation can then be neglected and reflection, from this point forward, will be understood to mean specular reflection.

Furthermore, throughout this report the following approximation will be used: Solar radiation will be considered to arive at a satellite surface as parallel rays, i.e., the sun will be treated as a point source at infinity.

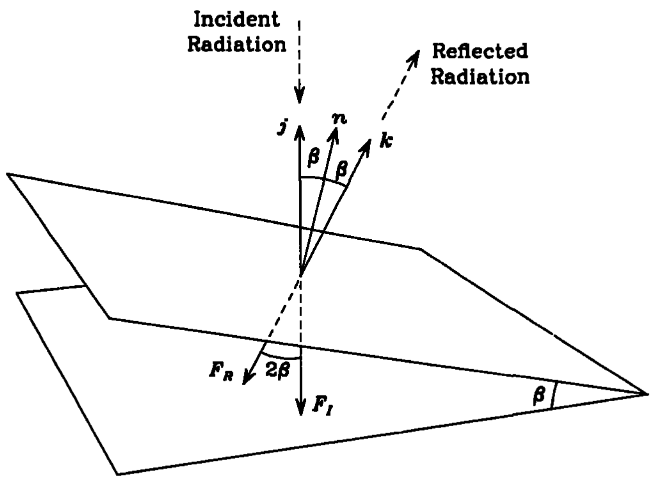

Figure 1.–Radiation forces on a flat plate.

Now we will consider the more general case of a flat plate inclined to the incident radiation as shown in figure 1. Let $\boldsymbol{j}$ be a unit vector pointing toward the sun and $\boldsymbol{n}$ a unit vector normal to the surface of the plate. If the area of the plate is A, then the projection of this area normal to the incident radiation is $A \cos \beta$, where $\cos \beta$ is the dot product of the unit vectors $\boldsymbol{j}$ and $\boldsymbol{n}$. Hence, the force imparted to the plate by incident solar radiation will be

$$\boldsymbol{F}_I = \frac{I}{c} A(\boldsymbol{j} \cdot \boldsymbol{n})(-\boldsymbol{j}). \tag{1}$$

A large fraction, R_S, of the incident radiation striking the plate will be reflected specularly in the direction of the unit vector $\boldsymbol{k}$. By Newton's third law this reflection will exert a force,

$$F_R = R_S \frac{I}{c} A(j \cdot n)(-k), \tag{2}$$

which acts in a direction opposite to that taken by the reflected radiation.

It is interesting to note that, if the plate is normal to the incident radiation, then j and k coincide, and the total force is the sum of F_I and F_R acting in the direction of $-j$. If R_S is close to unity, the total force can be nearly twice the incident force.

Any regular surface can be considered to be composed of an infinite number of flat plates of infinitesimal dimensions, each characterized by a unique pair of unit vectors n and k. The radiation forces acting on such a surface can be obtained by integrating the force equations, (1) and (2), over the sunlit portion of the surface. For a surface of revolution, such as a prolate spheroid, it is convenient to choose a coordinate system whose z-axis coincides with the axis of revolution, and define the surface by

$$x = u \cos \lambda, \qquad y = u \sin \lambda, \qquad \text{and} \qquad z = f(u), \tag{3}$$

using the polar coordinates u and λ.

The surface area of any figure of revolution can be obtained from (see, for example, Courant 1937)

$$A = \int_0^{2\pi} \int_0^{u_{max}} \left[1 + \left(\frac{dz}{du}\right)^2\right]^{1/2} u \, du \, d\lambda. \tag{4}$$

Substituting (4) into (1) and (2) provides the following general expressions for the forces exerted on surfaces of revolution by incident and reflected radiation:

$$F_I = -\frac{I}{c} j \int\int \left[1 + \left(\frac{dz}{du}\right)^2\right]^{1/2} (j \cdot n_{u,\lambda}) \, u \, du \, d\lambda, \tag{5}$$

and

$$F_R = -R_S \frac{I}{c} \int\int \left[1 + \left(\frac{dz}{du}\right)^2\right]^{1/2} (j \cdot n_{u,\lambda}) \, k_{u,\lambda} u \, du \, d\lambda, \tag{6}$$

where the integration is limited to the sunlit surface. The unit vectors n and k are subscripted u,λ in these expressions as a reminder that these vectors are functions of the integration variables.

RADIATION FORCES ON A SPHERE

It is a well known fact that the radiation force on a spherical satellite is equal to I/c multiplied by the cross-sectional area of the spherical surface. However, many users of this formula have never stopped to consider whether or not the force exerted by radiation reflected from the surface of the satellite is included in this expression. It is worthwhile, therefore, to consider the radiation forces acting on a sphere before attacking the more complex problem of a spheroidal surface.

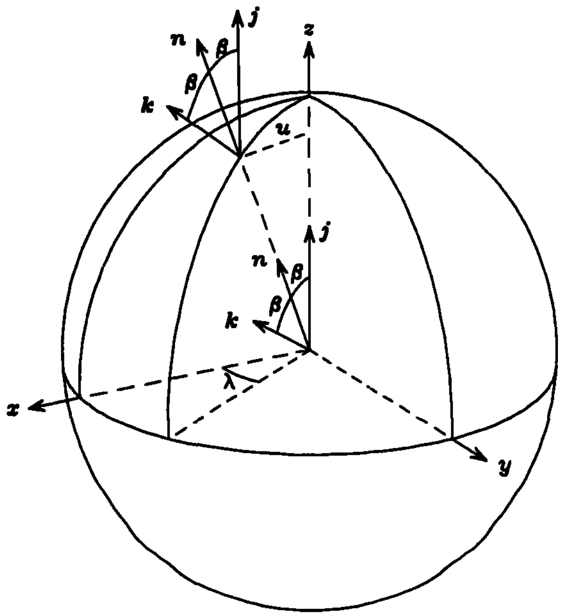

Figure 2.–Vectors associated with radiation forces at a point on the surface of a sphere.

For simplicity we choose a coordinate system with origin at the center of the sphere and z–axis pointing toward the sun. The orientation of the other two axes is immaterial. If r is the radius, the spherical surface is given by the parametric equations,

$$x = u \cos \lambda, \qquad y = u \sin \lambda, \qquad \text{and} \qquad z = (r^2 - u^2)^{1/2}. \tag{7}$$

Figure 2 shows the unit vectors associated with an arbitrary point on the surface. By our choice of coordinate system the unit vector j is parallel to the z–axis and, therefore,

$$j \cdot n_{u,\lambda} = \cos \beta = \frac{(r^2 - u^2)^{1/2}}{r}, \tag{8}$$

for all values of u and λ. From (7)

$$\left[1 + \left(\frac{dz}{du}\right)^2\right]^{1/2} = \frac{r}{(r^2 - u^2)^{1/2}}. \tag{9}$$

Hence, substituting (8) and (9) into (5) along with the appropriate limits we have

$$F_I = -\frac{I}{c} j \int_0^{2\pi} \int_0^r u \, du \, d\lambda = -\frac{I}{c} \pi r^2 j, \tag{10}$$

which is exactly I/c multiplied by the cross–sectional area. If the standard formula is correct, the reflected radiation force must be zero, as will now be shown.

From the figure it can be seen that the unit vector that defines the direction of the reflected radiation from any point on the surface is

$$k_{u,\lambda} = \begin{bmatrix} \sin 2\beta \cos \lambda \\ \sin 2\beta \sin \lambda \\ \cos 2\beta \end{bmatrix}. \tag{11}$$

Each component of the reflected force vector is obtained by substituting the corresponding component of k into (6). Since the x and y components of k contain $\cos \lambda$ and $\sin \lambda$, respectively, by inspection

$$F_{Rx} = -R_S \frac{I}{c} \int_0^{2\pi} \int_0^r u \sin 2\beta \cos \lambda \, du \, d\lambda = 0, \tag{12}$$

and

$$F_{Ry} = -R_S \frac{I}{c} \int_0^{2\pi} \int_0^r u \sin 2\beta \sin \lambda \, du \, d\lambda = 0, \tag{13}$$

as one might expect from consideration of symmetry. Using

$$\cos 2\beta = 1 - 2\sin^2 \beta = 1 - \frac{2u^2}{r^2},$$

the z component is

$$F_{Rz} = -R_S \frac{I}{c} \int_0^{2\pi} \int_0^r \left(u - \frac{2u^3}{r^2} \right) du \, d\lambda = 0. \tag{14}$$

Hence, for a spherical satellite the resultant force due to reflected radiation is zero, and the standard formula has been shown to be correct.

RADIATION FORCES ON A PROLATE SPHEROID

A prolate spheroid is generated by revolving an ellipse about its major axis. If the axis of revolution is chosen to be the z–axis and the semi–major and semi–minor axes of the ellipse are a and b, respectively, then the surface is given by the parametric equations

$$x = u \cos \lambda, \qquad y = u \sin \lambda, \qquad \text{and} \qquad z = \frac{a}{b}(b^2 - u^2)^{1/2}. \tag{15}$$

Differentiating we have

$$\frac{dz}{du} = - \frac{au}{b(b^2 - u^2)^{1/2}} \tag{16}$$

and

$$\left[1 + \left(\frac{dz}{du} \right)^2 \right]^{1/2} = \left[\frac{b^4 + a^2 e^2 u^2}{b^2 (b^2 - u^2)} \right]^{1/2}, \tag{17}$$

where e is the eccentricity of the spheroid.

The area of the surface can be obtained by a direct substitution of (17) into (4), but for reasons that will become apparent when the force integrals are developed, it is advantageous to use z rather than u as an integration variable. Solving (15) for u in terms of z provides

$$u = \frac{b}{a}(a^2-z^2)^{1/2},$$

from which we can obtain, substituting into (17),

$$\left[1+\left(\frac{dz}{du}\right)^2\right]^{1/2} = \frac{a(a^2-e^2z^2)^{1/2}}{bz} \tag{18}$$

and

$$u\,du = -\frac{b^2}{a^2}z\,dz. \tag{19}$$

Substituting (18) and (19) into (4) yields for the total surface area

$$A = -2\frac{b}{a}\int_0^{2\pi}\int_a^0 (a^2-e^2z^2)^{1/2}dz\,d\lambda,$$

which can be written

$$\begin{aligned} A &= \frac{b}{a}\int_0^{2\pi}\int_0^a (a^2-e^2z^2)^{1/2}dz\,d\lambda + \frac{b}{a}\int_0^{2\pi}\int_{-a}^0 (a^2-e^2z^2)^{1/2}dz\,d\lambda \\ &= (1-e^2)^{1/2}\int_0^{2\pi}\int_{-a}^{+a}(a^2-e^2z^2)^{1/2}dz\,d\lambda. \end{aligned} \tag{20}$$

Integration Limits

Derivation of the radiation forces exerted on a spheroid is considerably more complicated than for a sphere, because the cross-sectional area presented to the incoming radiation depends on the orientation of the unit vector j, which points toward the sun, to the major axis of the spheroid. For convenience we will choose a coordinate system in which z is the major axis of the spheroid, x is in the plane defined by z and j and forms an angle θ ($\leq\pi/2$) with j, and y completes a right-handed system. This coordinate system is illustrated in figure 3, which also shows the unit vectors that will be used in the derivation.

From the definition of the angle θ,

$$j = \begin{bmatrix} \cos\theta \\ 0 \\ \sin\theta \end{bmatrix}, \tag{21}$$

and from the figure it can be seen that the unit normal to the surface at any point forms an angle α with its projection parallel to the xy-plane. Therefore,

$$n_{z,\lambda} = \begin{bmatrix} \cos\alpha\cos\lambda \\ \cos\alpha\sin\lambda \\ \sin\alpha \end{bmatrix}. \tag{22}$$

Since the derivative dz/du, given by (16), is the slope of the tangent to the surface,

$$\tan\alpha = -\frac{du}{dz} = \frac{bz}{a(a^2-z^2)^{1/2}}, \tag{23}$$

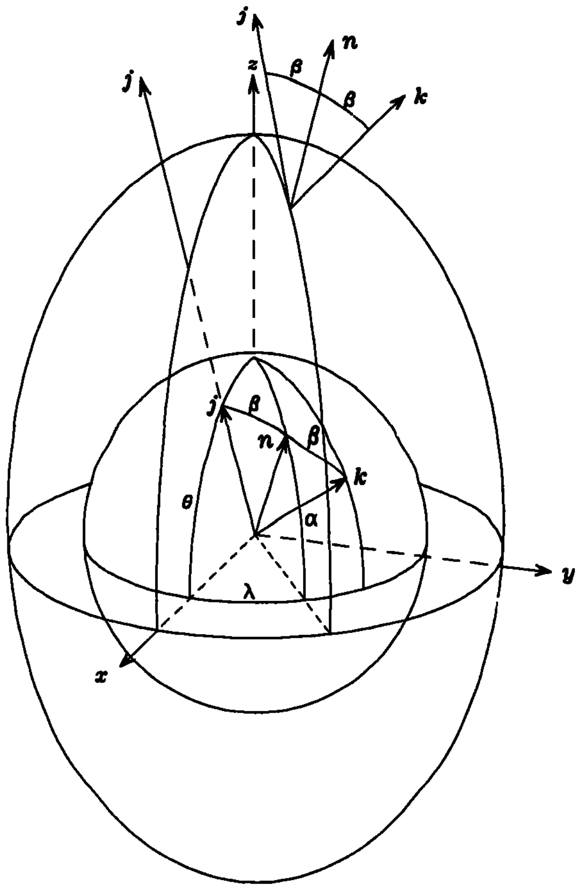

Figure 3.–Vectors associated with radiation forces at a point on the surface of a prolate spheroid.

which provides

$$\sin\alpha = \frac{z(1-e^2)^{1/2}}{(a^2-e^2z^2)^{1/2}} \qquad \text{and} \qquad \cos\alpha = \frac{(a^2-z^2)^{1/2}}{(a^2-e^2z^2)^{1/2}}. \tag{24}$$

Substituting (24) into (22) and forming the dot product with (21) obtains

$$j \cdot n_{z,\lambda} = \cos\beta = \frac{(a^2-z^2)^{1/2}\cos\theta\cos\lambda + z(1-e^2)^{1/2}\sin\theta}{(a^2-e^2z^2)^{1/2}}. \tag{25}$$

The radiation forces are obtained by integrating over the sunlit surface, whose boundary is an ellipse defined by the intersection of the spheroidal surface by the plane $j \cdot n_{z,\lambda} = 0$. As shown in figure 4, the boundary ellipse passes through the point $(z=\eta, \lambda=\pi)$ and all of the surface above the plane $z=\eta$ is in sunlight. The area of this portion of the surface can be obtained from

$$A_1 = (1-e^2)^{1/2}\int_{-\pi}^{+\pi}\int_{\eta}^{a}(a^2-e^2z^2)^{1/2}dz\, d\lambda. \tag{26}$$

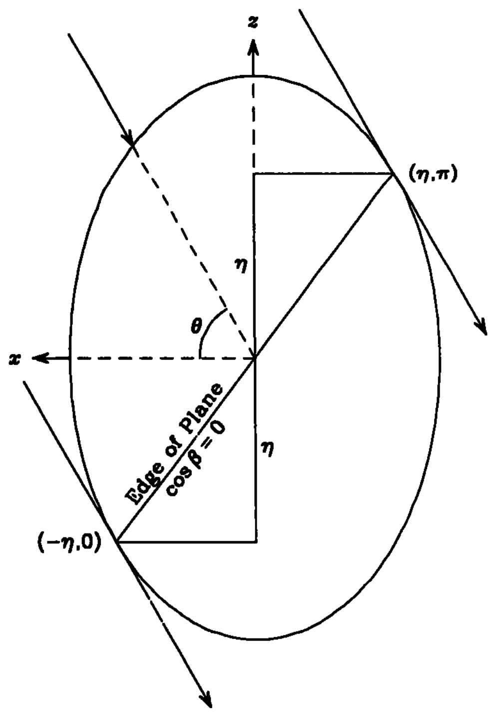

Figure 4a.–Cross–section of spheroid showing the intersection of the surface with the plane, $\cos\beta = 0$.

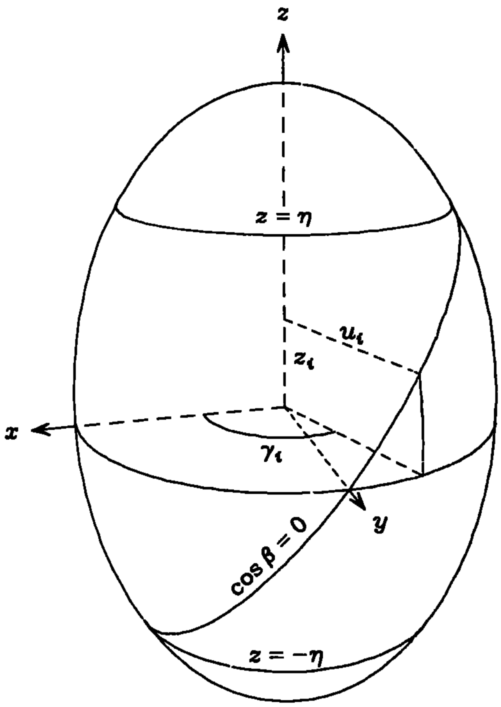

Figure 4b.–Geometrical significance of quantities used in defining the integration limits.

The boundary also passes through the point $(z=-\eta, \lambda=0)$ and none of the surface below the plane $z=-\eta$ is in sunlight. For each value of z, between the planes $z=\eta$ and $z=-\eta$, the boundary passes through a pair of points, $\lambda=\gamma(z)$ and $\lambda=-\gamma(z)$. Hence, the area of the sunlit surface between these planes is given by

$$A_2 = (1-e^2)^{1/2}\int_{-\gamma}^{+\gamma}\int_{-\eta}^{+\eta}(a^2-e^2z^2)^{1/2}dz\,d\lambda. \tag{27}$$

Setting $z=\eta$ and $\lambda=\pi$ in (25) and equating to zero obtains

$$\eta = \frac{a\cos\theta}{(1-e^2\sin^2\theta)^{1/2}}. \tag{28}$$

Setting $\lambda=\gamma$ in the same equation and again equating to zero we have

$$\cos\gamma = -\frac{z(1-e^2)^{1/2}\sin\theta}{(a^2-z^2)^{1/2}\cos\theta}, \tag{29}$$

and, since γ is less than or equal to $\pi/2$,

$$\sin\gamma = (1-\cos^2\gamma)^{1/2} = \frac{(1-e^2\sin^2\theta)^{1/2}(\eta^2-z^2)^{1/2}}{(a^2-z^2)^{1/2}\cos\theta}. \tag{30}$$

Hence, the total area of the sunlit surface is the sum of (26) and (27) where the integration limits are obtained from (28) and either (29) or (30).

However, throughout this section a number of integrals of the same form will be encountered. In order to simplify notation we will let S_1, operating on a function of the integration variables z and λ, represent the sum of double integrals:

$$S_1[f(z,\lambda)] = \int_{-\pi}^{+\pi}\int_{\eta}^{a} f(z,\lambda)\,dz\,d\lambda + \int_{-\gamma}^{+\gamma}\int_{-\eta}^{+\eta} f(z,\lambda)\,dz\,d\lambda. \tag{31}$$

In some instances the argument of this operator will be of the form $f(z)\cos\lambda$, in which case the first double integral will be zero. Therefore, extensive use will be made of a second operator,

$$S_2[f(z)] = \int_{-\eta}^{+\eta} f(z)\,dz, \tag{32}$$

to express the relationship:

$$S_1[f(z)\cos\lambda] = 2S_2[f(z)\sin\gamma]. \tag{33}$$

Using this notation, the area of the sunlit surface of the spheroid can be written

$$A = (1-e^2)^{1/2}S_1[(a^2-e^2z^2)^{1/2}] = (1-e^2)S_1[z\csc\alpha].$$

Then, using (5) and (6), the force equations become

$$F_I = -\frac{I}{c}(1-e^2)\,j\,S_1[z\csc\alpha\cos\beta] \tag{34}$$

and

$$F_R = -R_S\frac{I}{c}(1-e^2)\,S_1[z\,k_{z,\lambda}\csc\alpha\cos\beta], \tag{35}$$

and all that remains is to evaluate the necessary integrals.

The force of incident radiation impinging on the surface of a prolate spheroid is given by (34). Using (25), the operator in this equation can be written

$$S_1[z \csc\alpha \cos\beta] = \cos\theta\, S_1[z \cot\alpha \cos\lambda] + \sin\theta\, S_1[z]$$
$$= 2\cos\theta\, S_2[z \cot\alpha \sin\gamma] + \sin\theta\, S_1[z]. \tag{36}$$

The expression for $\sin\gamma$, equation (30), can be substituted into the argument of the first operator in (36) to produce a form that can be found in any book of integral tables. The second operator, however, represents the sum of two double integrals of which one is a standard form, but the other requires a rather lengthy solution. Therefore, a detailed procedure for evaluation of these operators is provided in Appendix 1, and only the results are presented here. They are:

$$S_1[z] = \frac{\pi ab \sin\theta}{(1-e^2\sin^2\theta)^{1/2}} \tag{37}$$

and

$$S_2[z \cot\alpha \sin\gamma] = \frac{\pi a^3 \cos\theta}{b(1-e^2\sin^2\theta)^{1/2}}. \tag{38}$$

Substituting these expressions into (36) and that result into (34) yields for the force of incident radiation:

$$F_I = -\frac{I}{c}\, j\, \pi ab(1-e^2\sin^2\theta)^{1/2}. \tag{39}$$

This equation states that the force of incident radiation is I/c multiplied by the area of the projection of the spheroid onto a plane normal to the incident radiation. This can be seen from figure 5. The ellipse formed by cutting the spheroidal surface by the plane $j \cdot n = 0$ will have semi-axes a' and b and will intersect the spheroidal surface at the point $(z = \eta, \lambda = \pi)$ as previously stated. The projection of this ellipse onto a plane normal to the radiation will be an ellipse with semi-axes a'' and b, where

$$a'' = a'\cos\phi \cos\alpha + a'\sin\phi \sin\alpha.$$

But

$$a'\sin\phi = \eta = \frac{a\cos\theta}{(1-e^2\sin^2\theta)^{1/2}},$$

$$a'\cos\phi = \frac{a(1-e^2)\sin\theta}{(1-e^2\sin^2\theta)^{1/2}},$$

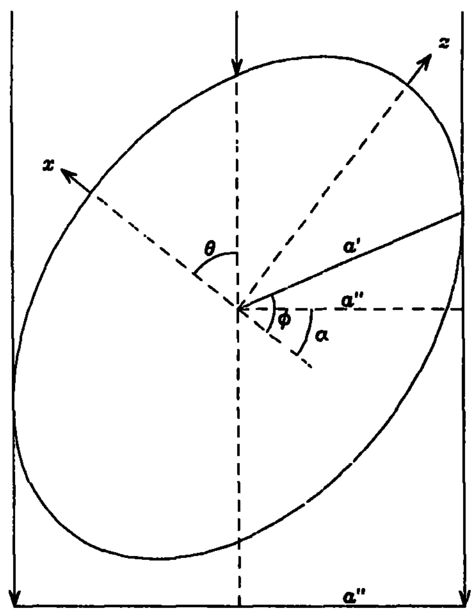

Figure 5.–Geometric determination of the force of incident radiation.

and from the figure it can be seen that $a = \pi/2 - \theta$. Hence

$$a'' = a(1-e^2\sin^2\theta)^{1/2},$$

which verifies that a purely geometrical approach will yield exactly the same result that was obtained by integration, i.e., equation (39).

Reflected Radiation Force

The force exerted on the spheroid by radiation reflected from its surface can be obtained from (35), but first the components of the unit vector $\boldsymbol{k}$ must be expressed in terms of the integration variables. Figure 6 shows a sphere whose center is located at an arbitrary point on the surface of the spheroid. The unit vector $\boldsymbol{j}$, as previously specified, lies in the xz-plane and forms an angle θ with the x-axis. The unit normal, $\boldsymbol{n}$, forms an angle α with its projection onto the xy-plane and an angle β with the vector $\boldsymbol{j}$. Then, by Snell's law, radiation striking that point on the surface will be reflected in the direction of $\boldsymbol{k}$, which lies on the great circle through $\boldsymbol{j}$ and $\boldsymbol{n}$ and forms an angle 2β with $\boldsymbol{j}$.

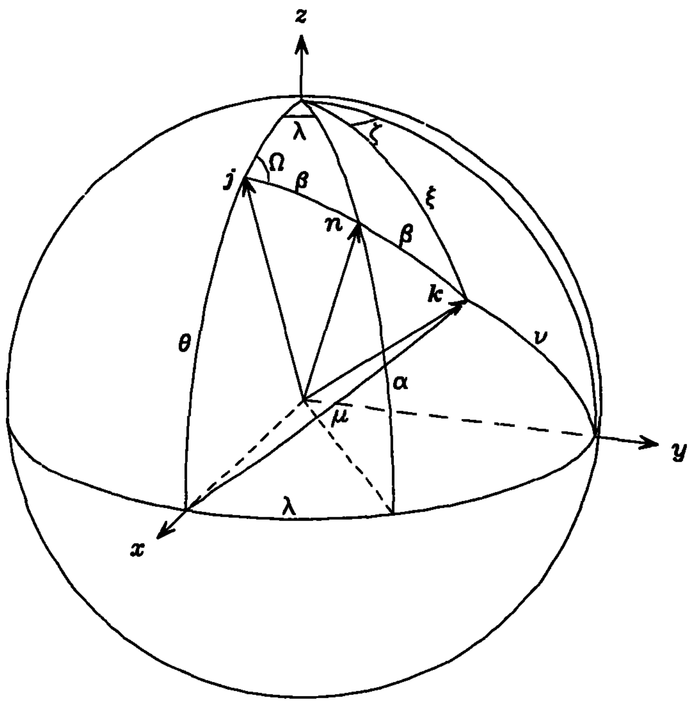

Figure 6.–Spherical triangles used in deriving the direction cosines of the unit vector $\boldsymbol{k}$.

From the spherical triangle formed by j, k, and the x-axis we have, by the cosine law,

$$\cos\mu = \cos\theta\cos 2\beta - \sin\theta\sin 2\beta\cos\Omega. \tag{40}$$

Also by the cosine law, using the spherical triangle formed by j, n, and the z-axis, we have

$$\cos\beta = \sin\theta\sin\alpha + \cos\theta\cos\alpha\cos\lambda \tag{41}$$

and

$$\cos\Omega = \frac{\sin\alpha - \sin\theta\cos\beta}{\cos\theta\sin\beta},$$

which, substituted into (41), after some manipulation yields

$$\cos\mu = \frac{2\cos\beta\,(\cos\beta - \sin\theta\sin\alpha)}{\cos\theta} - \cos\theta$$

$$= 2\cos\beta\cos\alpha\cos\lambda - \cos\theta. \tag{42}$$

Next, using the spherical triangle formed by j, k, and the z-axis, the law of cosines produces

$$\cos\xi = \sin\theta\cos 2\beta + \cos\theta\sin 2\beta\cos\Omega = 2\cos\beta\sin\alpha - \sin\theta. \tag{43}$$

The final component of k can be derived from the spherical triangle formed by k and the y and z-axes. Since one side of this triangle is $\pi/2$, the sine law yields

$$\cos\nu = \sin\xi\cos\zeta.$$

Using the sine law and the adjacent triangle formed by j, k, and the z-axis, we have

$$\cos\zeta = \frac{\sin 2\beta\sin\Omega}{\sin\xi}.$$

But

$$\sin\Omega = \frac{\cos\alpha\sin\lambda}{\sin\beta},$$

and, therefore,

$$\cos\nu = 2\cos\beta\cos\alpha\sin\lambda. \tag{44}$$

Collecting (42), (43), and (44), the unit vector becomes

$$k = \begin{bmatrix}\cos\mu\\ \cos\nu\\ \cos\xi\end{bmatrix} = \begin{bmatrix}2\cos\beta\cos\alpha\cos\lambda - \cos\theta\\ 2\cos\beta\cos\alpha\sin\lambda\\ 2\cos\beta\sin\alpha - \sin\theta\end{bmatrix}. \tag{45}$$

Substituting (45) into (35), the components of reflected force can be written

$$F_{Rx} = -R_S\frac{I}{c}(1-e^2)\,S_1[z\csc\alpha\cos\beta\cos\mu]$$

$$= -R_S\frac{I}{c}(1-e^2)\Big\{2\sin^2\theta\,S_1[z\sin\alpha\cos\alpha\cos\lambda]$$

$$+ 2\cos^2\theta\,S_1[z\cos^2\alpha\cot\alpha\cos^3\lambda] - \cos^2\theta\,S_1[z\cot\alpha\cos\lambda]$$

$$+ 4\sin\theta\cos\theta\,S_1[z\cos^2\alpha\cos^2\lambda] - \sin\theta\cos\theta\,S_1[z]\Big\},$$

$$F_{Ry} = -R_S\frac{I}{c}(1-e^2)\,S_1[z\csc\alpha\cos\beta\cos\nu]$$

$$= -R_S\frac{I}{c}(1-e^2)\Big\{2\sin^2\theta\,S_1[z\sin\alpha\cos\alpha\sin\lambda]$$

$$+ 2\cos^2\theta\,S_1[z\cos^2\alpha\cot\alpha\sin\lambda\cos^2\lambda]$$

$$+ 4\sin\theta\cos\theta\,S_1[z\cos^2\alpha\sin\lambda\cos\lambda]\Big\},$$

and

$$F_{Rz} = -R_S\frac{I}{c}(1-e^2)\,S_1[z\csc\alpha\cos\beta\cos\xi]$$

$$= -R_S\frac{I}{c}(1-e^2)\Big\{2\sin^2\theta\,S_1[z\sin^2\alpha] - \sin^2\theta\,S_1[z]$$

$$+ 2\cos^2\theta\,S_1[z\cos^2\alpha\cos^2\lambda] - \sin\theta\cos\theta\,S_1[z\cot\alpha\cos\lambda]$$

$$+ 4\sin\theta\cos\theta\,S_1[z\sin\alpha\cos\alpha\cos\lambda]\Big\}.$$

In these three equations the operator S_1 appears thirteen times with nine different arguments. Since S_1 represents the sum of two double integrals, it appears that we are faced with a long and tedious exercise in integral calculus. However, this task can be brought into manageable proportions by taking advantage of certain properties of the operator S_1 to simplify the above expressions. One of these properties has already been stated in (33), and the following list can be derived by integrating over λ:

$$S_1[f(z)\sin\lambda] = 0,$$

$$S_1[f(z)\sin\lambda\cos\lambda] = 0,$$

$$S_1[f(z)\sin\lambda\cos^2\lambda] = 0,$$

$$S_1[f(z)\cos^2\lambda] = \frac{1}{2}S_1[f(z)] + S_2[f(z)\sin\gamma\cos\gamma],$$

and

$$S_1[f(z)\cos^3\lambda] = \frac{4}{3}S_2[f(z)\sin\gamma] + \frac{2}{3}S_2[f(z)\sin\gamma\cos^2\gamma].$$

Furthermore, from (24) and (29) we have

$$\cos\gamma = -\tan\theta \tan\alpha,$$

which obtains

$$S_2[f(z)\sin\gamma\cos\gamma] = -\tan\theta\, S_2[f(z)\tan\alpha\sin\gamma]$$

and

$$S_2[f(z)\sin\gamma\cos^2\gamma] = \tan^2\theta\, S_2[f(z)\tan^2\alpha\sin\gamma].$$

Using these identities the components of reflected radiation force become

$$F_{Rx} = -R_S\frac{I}{c}(1-e^2)\Big\{4\sin^2\theta\, S_2[z\sin\alpha\cos\alpha\sin\gamma] - 2\cos^2\theta\, S_2[z\cot\alpha\sin\gamma]$$

$$+\frac{8}{3}\cos^2\theta\, S_2[z\cos^2\alpha\cot\alpha\sin\gamma] + \frac{4}{3}\sin^2\theta\, S_2[z\sin\alpha\cos\alpha\sin\gamma]$$

$$+ 2\sin\theta\cos\theta\, S_1[z\cos^2\alpha] - 4\sin^2\theta\, S_2[z\sin\alpha\cos\alpha\sin\gamma]$$

$$-\sin\theta\cos\theta\, S_1[z]\Big\},$$

$$F_{Ry} = 0,$$

and

$$F_{Rz} = -R_S\frac{I}{c}(1-e^2)\Big\{2\sin^2\theta\, S_1[z\sin^2\alpha] - \sin^2\theta\, S_1[z]$$

$$+\cos^2\theta\, S_1[z\cos^2\alpha] - 2\sin\theta\cos\theta\, S_2[z\sin\alpha\cos\alpha\sin\gamma]$$

$$-2\sin\theta\cos\theta\, S_2[z\cot\alpha\sin\gamma] + 8\sin\theta\cos\theta\, S_2[z\sin\alpha\cos\alpha\sin\gamma]\Big\}.$$

The number of different operator arguments can be further reduced by simplifying and collecting terms, after which the above equations become

$$F_{Rx} = -R_S\frac{I}{c}(1-e^2)\Big\{\sin\theta\cos\theta\, S_1[z] + \frac{2}{3}\cos^2\theta\, S_2[z\cot\alpha\sin\gamma]$$

$$+\frac{4}{3}(1-3\cos^2\theta)\, S_2[z\sin\alpha\cos\alpha\sin\gamma] - 2\sin\theta\cos\theta\, S_1[z\sin^2\alpha]\Big\}, \quad (46)$$

$$F_{Ry} = 0, \quad (47)$$

and

$$F_{Rz} = -R_S\frac{I}{c}(1-e^2)\Big\{(1-2\sin^2\theta)\, S_1[z] - 2\sin\theta\cos\theta\, S_2[z\cot\alpha\sin\gamma]$$

$$+ 6\sin\theta\cos\theta\, S_2[z\sin\alpha\cos\alpha\sin\gamma] - (1-3\sin^2\theta)\, S_1[z\sin^2\alpha]\Big\}. \quad (48)$$

Since the spheroid is symmetric about the xz-plane, which contains the satellite-sun vector, the reflected force perpendicular to this plane is zero as expected. The force components parallel to the x and z-axes are now functions of only four integral operators, of which two have already been encountered in deriving the incident force and are evaluated in Appendix 1. The remaining two operators represent integral forms of much greater complexity. A detailed procedure for their evaluation is provided in Appendix 2. The results obtained in Appendix 1 (written here in slightly different form) and Appendix 2 are:

$$S_1[z] = \frac{\pi a^2 U \sin\theta}{V},$$

$$S_2[z \cot\alpha \sin\gamma] = \frac{\pi a^2 \cos\theta}{2UV},$$

$$S_2[z \sin\alpha \cos\alpha \sin\gamma] = -\pi a^2\left(\frac{U\cos\theta}{2e^2V} + \frac{U^2-UV}{e^4\cos\theta}\right),$$

and (49)

$$S_1[z \sin^2\alpha] = -\pi a^2\left(\frac{U^3\sin\theta}{e^2V} + \frac{2U^2W}{e^4}\right),$$

where

$$U = (1-e^2)^{1/2},$$

$$V = (1-e^2\sin^2\theta)^{1/2},$$

and (50)

$$W = \ln\left(\frac{V+U\sin\theta}{1+\sin\theta}\right).$$

Substituting (49) into (46) through (48), the reflected force is given by

$$F_R = -R_S\frac{I}{c}\pi a^2\begin{bmatrix} P_x\cos\theta \\ 0 \\ P_z\sin\theta \end{bmatrix}, \tag{51}$$

where

$$\begin{aligned} P_x &= \frac{U^3\sin^2\theta}{V} + \frac{U\cos^2\theta}{3V} + \frac{2U^5\sin^2\theta}{e^2V} + \frac{4U^4W\sin\theta}{e^4} - \frac{2U^3}{3e^2V} \\ &\quad - \frac{4U^2(U^2-UV)}{3e^4\cos^2\theta} + \frac{2U^3\cos^2\theta}{e^2V} + \frac{4U^4}{e^4} - \frac{4U^3V}{e^4} \\ &= \frac{UV}{3} + \frac{4U^3V}{3e^2} - \frac{4U^3V}{e^4} - \frac{4U^2(U^2-UV)}{3e^4\cos^2\theta} + \frac{4U^4}{e^4}(1+W\sin\theta) \\ &= \frac{1}{e^4}\left[\left(-4+\frac{16}{3}e^2-e^4\right)UV - \frac{4U^2(U^2-UV)}{3\cos^2\theta} + 4U^4(1+W\sin\theta)\right] \end{aligned} \tag{52}$$

and

$$P_z = \frac{U^3}{V} - \frac{2U^3\sin^2\theta}{V} - \frac{U\cos^2\theta}{V} + \frac{U^5}{e^2V} + \frac{2U^4W}{e^4\sin\theta} - \frac{3U^5\sin^2\theta}{e^2V}$$
$$- \frac{6U^4W\sin\theta}{e^4} - \frac{3U^3\cos^2\theta}{e^2V} - \frac{6U^4}{e^4} + \frac{6U^3V}{e^4}$$

$$= -UV - \frac{2U^3V}{e^2} + \frac{6U^3V}{e^4} - \frac{6U^4}{e^4}(1+W\sin\theta) + \frac{2U^4W}{e^4\sin\theta}$$

$$= \frac{1}{e^4}\left[(6-8e^2+e^4)UV - 6U^4\left(1+W\sin\theta - \frac{W}{3\sin\theta}\right)\right]. \tag{53}$$

This completes the development of forces exerted on a prolate spheroid. Both the incident force, equation (38), and reflected force are expressed as functions of parameters that specify the size and shape of the spheroid and the direction of the sun. However, there are two disturbing features about the expression for reflected force. First, the presence of eccentricity raised to the fourth power in the denominators of (52) and (53) gives the impression that the reflected force will become infinitely large as the eccentricity of the spheroid approaches zero, i.e., if the surface is spherical. If this were true, disregarding the physical consequences, we would be faced with an inconsistency, because it was shown in an earlier section that the total force of radiation reflected from a spherical surface is equal to zero. This inconsistency does not exist, however, because it can be shown, using l'Hospital's rule, that both components of reflected force tend to zero as the eccentricity of the surface approaches zero. Furthermore, by the same technique, it can be shown that the unfortunate presence of $\cos\theta$ in the denominator of the middle term of (52) does not cause any problem, since the numerator of this term will go to zero as the denominator approaches zero.

Secondly, while equations (52) and (53) provide exact expressions for the components of reflected force, their complexity precludes any immediate insight into the nature of the force. It is, therefore, advantageous to develop series expansions for these expressions. This will serve three purposes: (1) The magnitudes, and even the directions, of the force components are much more easily appreciated from the series representations, (2) From the series expansions it can readily be seen that the limits of these force components do indeed go to zero as the spheroidal eccentricity approaches zero, and (3) When we treat the case of a spheroid rotating about a minor axis, in the next section, it will be necessary to use series, because the forces are not representable in closed form.

Series Development

P_x and P_z can be developed as a power series in the small parameter e^2. If the series are desired to the ith-order in e^2, it is neccessary to carry terms to the order $i+2$ in developing series for the expressions enclosed in brackets in (52) and (53), because both of these quantities will be divided by e^4. Due to space limitations the procedure will be illustrated by developing only the first three terms of the series expressions for P_x and P_z. We begin by writing U and V as the power series

$$U = 1 - \frac{e^2}{2} - \frac{e^4}{8} - \frac{e^6}{16} - \cdots \tag{54}$$

and

$$V = 1 - \frac{e^2}{2}\sin^2\theta - \frac{e^4}{8}\sin^4\theta - \frac{e^6}{16}\sin^6\theta - \cdots \tag{55}$$

The product UV, which appears in both (52) and (53), is obtained by direct multiplication of these series to form

$$\begin{aligned} UV = 1 &- \frac{e^2}{2}(1+\sin^2\theta) - \frac{e^4}{8}(1-2\sin^2\theta + \sin^4\theta) \\ &- \frac{e^6}{16}(1-\sin^2\theta - \sin^4\theta + \sin^6\theta) - \cdots. \end{aligned} \tag{56}$$

If we subtract (56) from $U^2 = 1-e^2$, we have a series in which all terms are divisible by $1-\sin^2\theta$. After division we have

$$\frac{U^2-UV}{\cos^2\theta} = -\frac{e^2}{2} + \frac{e^4}{8}(1-\sin^2\theta) + \frac{e^6}{16}(1-\sin^4\theta) + \cdots, \tag{57}$$

which will be required for the middle term of (52).

Now, if (54) is multiplied by $\sin\theta$ and added to (55), the resulting series is exactly divisible by $1+\sin\theta$. Hence,

$$\begin{aligned} Q &= \frac{V+U\sin\theta}{1+\sin\theta} \\ &= 1 - \frac{e^2}{2}\sin\theta - \frac{e^4}{8}(\sin\theta - \sin^2\theta + \sin^3\theta) \\ &\quad - \frac{e^6}{16}(\sin\theta - \sin^2\theta + \sin^3\theta - \sin^4\theta + \sin^5\theta) - \cdots. \end{aligned}$$

From this equation it is easy to see that when $\theta = 0$, $Q = V = 1$, and when $\theta = \pi/2$, $Q = U$. Therefore, $0 < Q \leqq 1$, and we can use

$$W = \ln Q = (Q-1) - \frac{1}{2}(Q-1)^2 + \frac{1}{3}(Q-1)^3 - \cdots$$

which obtains

$$W = \sin\theta\left[-\frac{e^2}{2} - \frac{e^4}{8}(1+\sin^2\theta) - \frac{e^6}{16}(1+\frac{2}{3}\sin^2\theta + \sin^4\theta) - \cdots\right]. \tag{58}$$

Since the series for W contains the factor $\sin\theta$, then both $1+W\sin\theta$ and $W/\sin\theta$, as well as (56) and (57), are power series in e^2 in which the coeffieients are polynomials in $\sin^2\theta$. Furthermore, the degree of these polynomial coefficients is less than, or equal to, the power to which e^2 is raised in the term in which they occur. Therefore, it is merely a matter of bookkeeping to perform the multiplications and additions indicated in (52) and (53). One method of handling this bookkeeping is shown in Tables 1 and 2.

From the tables it can be seen that the coefficients of all terms in e^0, e^2, and e^4 are zero so that, after division by e^4, only terms in e^2 and higher powers of e remain. By carrying along higher-order terms it can be shown that both P_x and P_z are representable by power series in e^2 in which the coefficients of the ith-order terms are polynomials of degree i in $\sin^2\theta$. The first three terms of these series are

$$P_x = \frac{e^2}{6}(1+\sin^2\theta) - \frac{e^4}{48}(1+10\sin^2\theta - 3\sin^2\theta)$$
$$- \frac{e^6}{48}(1+\sin^2\theta + 3\sin^4\theta - \frac{9}{5}\sin^6\theta) - \cdots \quad (59)$$

and

$$P_z = -\frac{e^2}{6}(3-\sin^2\theta) + \frac{e^4}{48}(3+2\sin^2\theta + 3\sin^4\theta)$$
$$+ \frac{e^6}{48}(3-\sin^2\theta - \frac{3}{5}\sin^4\theta + \frac{9}{5}\sin^6\theta) + \cdots. \quad (60)$$

It is immediately apparent that both P_x and P_z are zero when e is zero. Therefore, the reflected force, given by (51), is consistent with the results previously obtained. When e is greater than zero, P_x is greater than zero and P_z is less than zero for all values of θ. Hence, when $\theta = 0$, $F_{Rz} = 0$ and F_{Rx} is negative. The reflected force and the incident force both act in the direction of the negative x-axis, and the approximation used by Smith and Fea (1970) for finding the direction of the resultant reflected force obtains the correct result. However, when $\theta = \pi/2$, $F_{Rx} = 0$ and F_{Rz} is positive. In this case the reflected force acts in the direction of the positive z-axis, while the incident force is directed along the negative z-axis. This is in direct contradiction to the approximation of Smith and Fea, because the ray passing through the center of the spheroid would be reflected back along the positive z-axis and, therefore, the reflected force would be expected to act in the direction of the negative z-axis.

In fact it can be shown, using either the exact expressions (52) and (53) or the series representations (59) and (60), that the approximation of Smith and Fea does not obtain the correct direction for the reflected force except when θ is equal to zero. However, since the incident radiation force is so much larger than the reflected force, the resultant will be in error by only about 20% in the worst case. This discrepancy is considerably less for a spheroid which is rotating about a minor axis, as will be seen in the next section.

In summary, the force of incident radiation on a stationary, prolate spheroidal satellite is given by

$$F_I = -\frac{I}{c}\pi a^2 UV \begin{bmatrix} \cos\theta \\ 0 \\ \sin\theta \end{bmatrix}, \quad (61)$$

Table 1.—Series development of P_x.

Series Used in Developing P_x:

$$UV = 1 - \frac{1}{2}e^2 - \frac{1}{2}e^2\sin^2\theta - \frac{1}{8}e^4 + \frac{1}{4}e^4\sin^2\theta - \frac{1}{8}e^4\sin^4\theta - \frac{1}{16}e^6 + \frac{1}{16}e^6\sin^2\theta + \frac{1}{16}e^6\sin^4\theta - \frac{1}{16}e^6\sin^6\theta - \frac{5}{128}e^8 + \frac{1}{32}e^8\sin^2\theta + \frac{1}{64}e^8\sin^4\theta + \frac{1}{32}e^8\sin^6\theta - \frac{5}{128}e^8\sin^8\theta$$

$$\frac{U^2-UV}{\cos^2\theta} = -\frac{1}{2}e^2 + \frac{1}{8}e^4 - \frac{1}{8}e^4\sin^2\theta + \frac{1}{16}e^6 - \frac{1}{16}e^6\sin^4\theta + \frac{5}{128}e^8 + \frac{1}{128}e^8\sin^2\theta - \frac{1}{128}e^8\sin^4\theta - \frac{5}{128}e^8\sin^6\theta$$

$$1+W\sin\theta = 1 - \frac{1}{2}e^2\sin^2\theta - \frac{1}{8}e^4\sin^2\theta - \frac{1}{8}e^4\sin^4\theta - \frac{1}{16}e^6\sin^2\theta - \frac{1}{24}e^6\sin^4\theta - \frac{1}{16}e^6\sin^6\theta - \frac{5}{128}e^8\sin^2\theta - \frac{3}{128}e^8\sin^4\theta - \frac{3}{128}e^8\sin^6\theta - \frac{5}{128}e^8\sin^8\theta$$

Summation to Obtain Series for $e^4P_x = (-4+\frac{16}{3}e^2-e^4)UV - \frac{4}{3}(1-e^2)\frac{U^2-UV}{\cos^2\theta} + (4-8e^2+4e^4)(1+W\sin\theta)$:

$$-4UV = -4 + 2e^2 + 2e^2\sin^2\theta + \frac{1}{2}e^4 - e^4\sin^2\theta + \frac{1}{2}e^4\sin^4\theta + \frac{1}{4}e^6 - \frac{1}{4}e^6\sin^2\theta - \frac{1}{4}e^6\sin^4\theta + \frac{1}{4}e^6\sin^6\theta + \frac{5}{32}e^8 - \frac{1}{8}e^8\sin^2\theta - \frac{1}{16}e^8\sin^4\theta - \frac{1}{8}e^8\sin^6\theta + \frac{5}{32}e^8\sin^8\theta$$

$$+\frac{16}{3}e^2UV = +\frac{16}{3}e^2 - \frac{8}{3}e^4 - \frac{8}{3}e^4\sin^2\theta - \frac{2}{3}e^6 + \frac{4}{3}e^6\sin^2\theta - \frac{2}{3}e^6\sin^4\theta - \frac{1}{3}e^8 + \frac{1}{3}e^8\sin^2\theta + \frac{1}{3}e^8\sin^4\theta - \frac{1}{3}e^8\sin^6\theta$$

$$-e^4UV = -e^4 + \frac{1}{2}e^6 + \frac{1}{2}e^6\sin^2\theta + \frac{1}{8}e^8 - \frac{1}{4}e^8\sin^2\theta + \frac{1}{8}e^8\sin^4\theta$$

$$-\frac{4}{3}\frac{U^2-UV}{\cos^2\theta} = +\frac{2}{3}e^2 - \frac{1}{6}e^4 + \frac{1}{6}e^4\sin^2\theta - \frac{1}{12}e^6 + \frac{1}{12}e^6\sin^4\theta - \frac{5}{96}e^8 - \frac{1}{96}e^8\sin^2\theta + \frac{1}{96}e^8\sin^4\theta + \frac{5}{96}e^8\sin^6\theta$$

$$+\frac{4}{3}e^2\frac{U^2-UV}{\cos^2\theta} = -\frac{2}{3}e^4 + \frac{1}{6}e^6 - \frac{1}{6}e^6\sin^2\theta + \frac{1}{12}e^8 - \frac{1}{12}e^8\sin^4\theta$$

$$+4(1+W\sin\theta) = +4 - 2e^2\sin^2\theta - \frac{1}{2}e^4\sin^2\theta - \frac{1}{2}e^4\sin^4\theta - \frac{1}{4}e^6\sin^2\theta - \frac{1}{6}e^6\sin^4\theta - \frac{1}{4}e^6\sin^6\theta - \frac{5}{32}e^8\sin^2\theta - \frac{3}{32}e^8\sin^4\theta - \frac{3}{32}e^8\sin^6\theta - \frac{5}{32}e^8\sin^8\theta$$

$$-8e^2(1+W\sin\theta) = -8e^2 + 4e^4\sin^2\theta + e^6\sin^2\theta + e^6\sin^4\theta + \frac{1}{2}e^8\sin^2\theta + \frac{1}{3}e^8\sin^4\theta + \frac{1}{2}e^8\sin^6\theta$$

$$+4e^4(1+W\sin\theta) = +4e^4 - 2e^6\sin^2\theta - \frac{1}{2}e^8\sin^2\theta - \frac{1}{2}e^8\sin^4\theta$$

$$e^4P_x = +\frac{1}{6}e^6 + \frac{1}{6}e^6\sin^2\theta - \frac{1}{48}e^8 - \frac{5}{24}e^8\sin^2\theta + \frac{1}{16}e^8\sin^4\theta$$

Table 2.--Series development of P_z.

Series Used in Developing P_z:

$$UV = 1 - \frac{1}{2}e^2 - \frac{1}{2}e^2\sin^2\theta - \frac{1}{8}e^4 + \frac{1}{4}e^4\sin^2\theta - \frac{1}{8}e^4\sin^4\theta - \frac{1}{16}e^6 + \frac{1}{16}e^6\sin^2\theta + \frac{1}{16}e^6\sin^4\theta - \frac{1}{16}e^6\sin^6\theta - \frac{5}{128}e^8 + \frac{1}{32}e^8\sin^2\theta + \frac{1}{64}e^8\sin^4\theta + \frac{1}{32}e^8\sin^6\theta - \frac{5}{128}e^8\sin^8\theta$$

$$1+W\sin\theta = 1 - \frac{1}{2}e^2\sin^2\theta - \frac{1}{8}e^4\sin^2\theta - \frac{1}{8}e^4\sin^4\theta - \frac{1}{16}e^6\sin^2\theta - \frac{1}{24}e^6\sin^4\theta - \frac{1}{16}e^6\sin^6\theta - \frac{5}{128}e^8\sin^2\theta - \frac{3}{128}e^8\sin^4\theta - \frac{3}{128}e^8\sin^6\theta - \frac{5}{128}e^8\sin^8\theta$$

$$\frac{W}{\sin\theta} = -\frac{1}{2}e^2 - \frac{1}{8}e^4 - \frac{1}{8}e^4\sin^2\theta - \frac{1}{16}e^6 - \frac{1}{24}e^6\sin^2\theta - \frac{1}{16}e^6\sin^4\theta - \frac{5}{128}e^8 - \frac{3}{128}e^8\sin^2\theta - \frac{3}{128}e^8\sin^4\theta - \frac{5}{128}e^8\sin^6\theta$$

Summation to Obtain Series for $e^4P_z = (6-8e^2+e^4)UV + (-6+12e^2-6e^4)(1+W\sin\theta) + (2-4e^2+2e^4)\dfrac{W}{\sin\theta}$:

$$6UV = 6 - 3e^2 - 3e^2\sin^2\theta - \frac{3}{4}e^4 + \frac{3}{2}e^4\sin^2\theta - \frac{3}{4}e^4\sin^4\theta - \frac{3}{8}e^6 + \frac{3}{8}e^6\sin^2\theta + \frac{3}{8}e^6\sin^4\theta - \frac{3}{8}e^6\sin^6\theta - \frac{15}{64}e^8 + \frac{3}{16}e^8\sin^2\theta + \frac{3}{32}e^8\sin^4\theta + \frac{3}{16}e^8\sin^6\theta - \frac{15}{64}e^8\sin^8\theta$$

$$-8e^2UV = -8e^2 + 4e^4 + 4e^4\sin^2\theta + e^6 - 2e^6\sin^2\theta + e^6\sin^4\theta + \frac{1}{2}e^8 - \frac{1}{2}e^8\sin^2\theta - \frac{1}{2}e^8\sin^4\theta + \frac{1}{2}e^8\sin^6\theta$$

$$+e^4UV = + e^4 - \frac{1}{2}e^6 - \frac{1}{2}e^6\sin^2\theta - \frac{1}{8}e^8 + \frac{1}{4}e^8\sin^2\theta - \frac{1}{8}e^8\sin^4\theta$$

$$-6(1+W\sin\theta) = -6 + 3e^2\sin^2\theta + \frac{3}{4}e^4\sin^2\theta + \frac{3}{4}e^4\sin^4\theta + \frac{3}{8}e^6\sin^2\theta + \frac{1}{4}e^6\sin^4\theta + \frac{3}{8}e^6\sin^6\theta + \frac{15}{64}e^8\sin^2\theta + \frac{9}{64}e^8\sin^4\theta + \frac{9}{64}e^8\sin^6\theta + \frac{15}{64}e^8\sin^8\theta$$

$$+12e^2(1+W\sin\theta) = +12e^2 - 6e^4\sin^2\theta - \frac{3}{2}e^6\sin^2\theta - \frac{3}{2}e^6\sin^4\theta - \frac{3}{4}e^8\sin^2\theta - \frac{1}{2}e^8\sin^4\theta - \frac{3}{4}e^8\sin^6\theta$$

$$-6e^4(1+W\sin\theta) = -6e^4 + 3e^6\sin^2\theta + \frac{3}{4}e^8\sin^2\theta + \frac{3}{4}e^8\sin^4\theta$$

$$+2\frac{W}{\sin\theta} = -e^2 - \frac{1}{4}e^4 - \frac{1}{4}e^4\sin^2\theta - \frac{1}{8}e^6 - \frac{1}{12}e^6\sin^2\theta - \frac{1}{8}e^6\sin^4\theta - \frac{5}{64}e^8 - \frac{3}{64}e^8\sin^2\theta - \frac{3}{64}e^8\sin^4\theta - \frac{5}{64}e^8\sin^6\theta$$

$$-4e^2\frac{W}{\sin\theta} = +2e^4 + \frac{1}{2}e^6 + \frac{1}{2}e^6\sin^2\theta + \frac{1}{4}e^8 + \frac{1}{6}e^8\sin^2\theta + \frac{1}{4}e^8\sin^4\theta$$

$$+2e^4\frac{W}{\sin\theta} = -e^6 - \frac{1}{4}e^8 - \frac{1}{4}e^8\sin^2\theta$$

$$e^4P_z = -\frac{1}{2}e^6 + \frac{1}{6}e^6\sin^2\theta + \frac{1}{16}e^8 + \frac{1}{24}e^8\sin^2\theta + \frac{1}{16}e^8\sin^4\theta$$

where U and V are given by (50). The force due to radiation reflected from the surface of the satellite is

$$F_R = -R_S \frac{I}{c} \pi a^2 \begin{bmatrix} P_x \cos\theta \\ 0 \\ P_z \sin\theta \end{bmatrix}, \tag{62}$$

where P_x and P_z are given exactly by (52) and (53) or by the series (59) and (60).

RADIATION FORCES ON A ROTATING PROLATE SPHEROID

In the last section expressions were developed for the forces exerted on a stationary prolate spheroid. These equations were developed in a coordinate system in which the z-axis is the major axis of the spheroid and the x-axis is in the plane defined by the z-axis and the sun. If the spheroid rotates about its major axis, there is no reason why this coordinate system cannot remain fixed in space. Therefore, the force equations given for a stationary spheroid can be used in the case of rotation about the major axis. They will, however, be written in a form that is better adapted to perturbation equations. When the spheroid rotates about a minor axis, however, the xz-plane, which was supposed to contain the satellite-sun vector will rotate away from that vector, and the assumptions on which the force equations of the previous section were based no longer hold. Obviously, for this case, new force equations must be derived, but first we will treat the case of rotation about the major axis.

Rotation About the Major Axis

In developing perturbation formulas for solar radiation it has been common practice to express the radiation force in the (X,Y,Z) coordinate system where, X is the direction of the sun, Z is the direction of the pole of the ecliptic, and Y completes a right-handed system. This is a logical choice because forces given in this system can be expressed in satellite orbit coordinates by a standard transformation involving the ecliptic longitude of the sun λ, obliquity of the ecliptic ϵ, right ascension of the ascending node, inclination of the satellite orbit, and argument of perigee.

Referring to figure 7 it can be seen that any vector in the (x,y,z) coordinate system, which was used for developing the force equations, can be transformed into the (X,Y,Z) system in two steps. First, a negative rotation about the y-axis through the angle θ brings us into the intermediate system (X',Y',Z'), i.e.,

$$\begin{bmatrix} X' \\ Y' \\ Z' \end{bmatrix} = \mathcal{R}_2(-\theta) \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x\cos\theta + z\sin\theta \\ y \\ -x\sin\theta + z\cos\theta \end{bmatrix}. \tag{63}$$

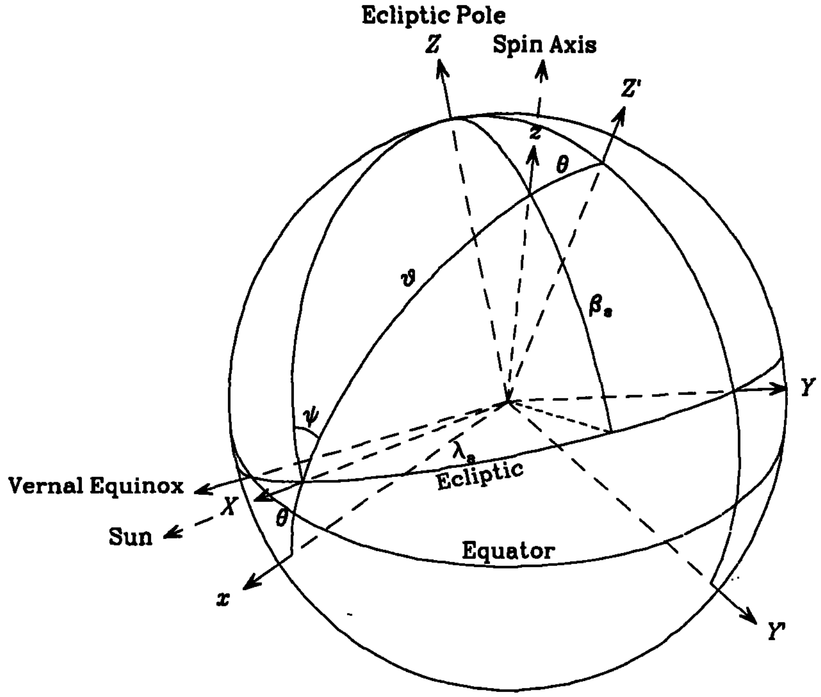

Figure 7.–Coordinate systems used in describing the radiation forces on a prolate spheroid rotating about its major axis.

Then, a positive rotation about the X'–axis through an angle ψ,

$$\begin{bmatrix} X \\ Y \\ Z \end{bmatrix} = \mathcal{R}_1(\psi) \begin{bmatrix} X' \\ Y' \\ Z' \end{bmatrix} = \begin{bmatrix} X' \\ Y'\cos\psi + Z'\sin\psi \\ -Y'\sin\psi + Z'\cos\psi \end{bmatrix}, \tag{64}$$

completes the transformation provided that θ and ψ can be obtained in terms of known quantities.

Assume that the position of the spin axis, in this case the z–axis, has been determined by some means, perhaps photoelectric photometry. If the spin axis position is given in terms of ecliptic latitude β_s and ecliptic longitude λ_s, then from the figure and a little spherical trigonometry we can obtain

$$\sin\psi = \frac{\cos\beta_s \sin\lambda_s \cos\lambda - \cos\beta_s \cos\lambda_s \sin\lambda}{\cos\theta},$$

$$\cos\psi = \frac{\sin\beta_s}{\cos\theta},$$

and (65)

$$\sin\theta = \cos\vartheta = \cos\beta_s \cos\lambda_s \cos\lambda + \cos\beta_s \sin\lambda_s \sin\lambda,$$

where $\cos\theta = \sin\vartheta$ can be obtained from its co–function, since θ is by definition less than or equal to $\pi/2$.

Of course it is unlikely that the position of the spin axis will be given in the ecliptic system. But, if the right ascension α_s and declination δ_s of the spin axis are known, we can use the standard transformation given in any astronomy text,

$$\cos\beta_s \cos\lambda_s = \cos\delta_s \cos\alpha_s,$$

$$\cos\beta_s \sin\lambda_s = \cos\delta_s \sin\alpha_s \cos\varepsilon + \sin\delta_s \sin\varepsilon,$$

and (66)

$$\sin\beta_s = -\cos\delta_s \sin\alpha_s \sin\varepsilon + \sin\delta_s \cos\varepsilon,$$

which can be substituted into (65) to obtain the required rotation components in terms of known angles.

The force of incident radiation can now be expressed in the (X,Y,Z) system by applying the transformations (63) and (64) to equation (61). However, after applying the first rotation it is found that the Y' and Z' components are both zero so that the second rotation is inconsequential. Hence,

$$F_I = -\frac{I}{c}\pi a^2 P_I \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \tag{67}$$

where $P_I = UV$. Applying these transformations to (62) we find that the reflected radiation force will be

$$F_R = -R_S\frac{I}{c}\pi a^2 \begin{bmatrix} P_X \\ P_{Z'}\sin\psi \\ P_{Z'}\cos\psi \end{bmatrix}, \tag{68}$$

where

$$P_X = P_x\cos^2\theta + P_z\sin^2\theta$$

and

$$P_{Z'} = (P_z - P_x)\sin\theta\cos\theta.$$

The forces are now given in the desired coordinate system, but for consistency with the force equations that will be developed in the next subsection, it is preferable to express the forces as functions of ϑ, the angle between the sun and the spin axis. Using (59) and (60), P_X and $P_{Z'}$ can be obtained as functions of θ and then, since ϑ is the complement of θ, we have

$$P_I = (1-e^2)^{1/2}(1-e^2\cos^2\vartheta)^{1/2}, \tag{69}$$

$$P_X = -\frac{e^2}{3}(1 - \frac{3}{2}\sin^2\vartheta) + \frac{e^4}{6}(1 - 3\sin^2\vartheta + \frac{15}{8}\sin^4\vartheta)$$
$$+ \frac{e^6}{15}(1 - 3\sin^2\vartheta + 3\sin^4\vartheta - \frac{21}{16}\sin^6\vartheta) + \cdots, \tag{70}$$

and

$$P_{Z'} = 2\sin\vartheta\cos\vartheta\left[-\frac{e^2}{3} + \frac{e^4}{6}(1 - \frac{3}{4}\sin^2\vartheta) + \frac{e^6}{15}(1 - \frac{3}{4}\sin^2\vartheta + \frac{3}{8}\sin^4\vartheta) + \cdots\right]. \tag{71}$$

Then the radiation forces on a prolate spheroid that is rotating about its major axis are (67) and (68) where the P's are given by (69)through (71) and ψ and ϑ are obtained from (65).

Rotation About a Minor Axis

First, assume that the spheroidal satellite is stationary and define the coordinates (x,y,z) as before. The vector j, as before, forms an angle θ with the x-axis and lies in the xz-plane, where z coincides with the major axis of the spheroid. Let the x-axis be the spin axis and redefine the coordinate system such that the z-axis is in the plane defined by j and the spin axis. By definition, θ is the angle between j and the spin axis and z is no longer tied to the major axis of the spheroid. This redefinition does not change anything as long as the spheroid remains stationary. However, after the spheroid has rotated through some angle ω, as shown in figure 8, the major axis no longer coincides with z, but has rotated to z'.

Let us now define an instantaneous coordinate system (x',y',z') such that z' is the major axis of the spheroid, the x'-axis is in the plane defined by z' and j, and the y'-axis completes a right-handed system. This is the same coordinate system used for deriving the force equations for a stationary spheroid, except that the angle between x' and j is θ'. Therefore, (61) and (62) provide the incident and reflected forces, respectively, in the (x',y',z') system if θ is replaced by θ' wherever it appears in either of these equations.

From the figure it can be seen that

$$\sin\theta' = \sin\vartheta\cos\omega \quad \text{and} \quad \cos\theta' = \frac{\cos\vartheta}{\cos\lambda'}.$$

Substituting these expressions for $\sin\theta$ and $\cos\theta$, respectively, in (61) and (62) yields

$$\begin{bmatrix} F_{Ix'} \\ F_{Iy'} \\ F_{Iz'} \end{bmatrix} = -\frac{I}{c}\pi a^2 P_{I'} \begin{bmatrix} \dfrac{\cos\vartheta}{\cos\lambda'} \\ 0 \\ \sin\vartheta\cos\omega \end{bmatrix} \tag{72}$$

and

$$\begin{bmatrix} F_{Rx'} \\ F_{Ry'} \\ F_{Rz'} \end{bmatrix} = -R_S\frac{I}{c}\pi a^2 \begin{bmatrix} P_{x'}\dfrac{\cos\vartheta}{\cos\lambda'} \\ 0 \\ P_{z'}\sin\vartheta\cos\omega \end{bmatrix}, \tag{73}$$

where

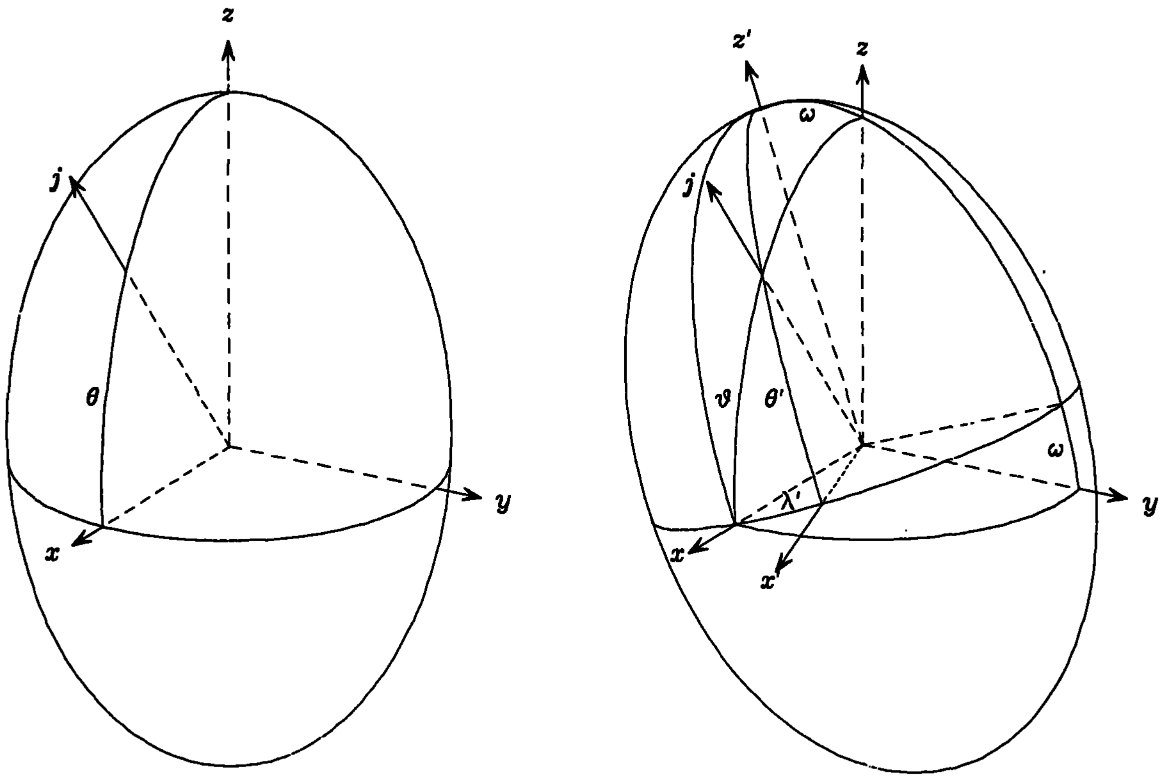

Figure 8.–Prolate spheroid in its initial position and after a rotation about the x–axis through an angle ω.

$$P_{I'} = 1 - \frac{e^2}{2}(1 + \sin^2\vartheta \cos^2\omega) - \frac{e^4}{8}(1 - 2\sin^2\vartheta \cos^2\omega + \sin^4\vartheta \cos^4\omega)$$
$$- \frac{e^6}{16}(1 - \sin^2\vartheta \cos^2\omega - \sin^4\vartheta \cos^4\omega + \sin^6\vartheta \cos^6\omega) - \cdots,$$

$$P_{x'} = \frac{e^2}{6}(1 + \sin^2\vartheta \cos^2\omega) - \frac{e^4}{48}(1 + 10\sin^2\vartheta \cos^2\omega - 3\sin^4\vartheta \cos^4\omega)$$
$$- \frac{e^6}{48}(1 + \sin^2\vartheta \cos^2\omega + 3\sin^4\vartheta \cos^4\omega - \frac{9}{5}\sin^6\vartheta \cos^6\omega) - \cdots,$$

and (74)

$$P_{z'} = \frac{e^2}{6}(3 - \sin^2\vartheta \cos^2\omega) + \frac{e^4}{48}(3 + 2\sin^2\vartheta \cos^2\omega + 3\sin^4\vartheta \cos^4\omega)$$
$$+ \frac{e^6}{48}(3 - \sin^2\vartheta \cos^2\omega - \frac{3}{5}\sin^4\vartheta \cos^4\omega + \frac{9}{5}\sin^6\vartheta \cos^6\omega) + \cdots.$$

We can now transform the force vectors into the fixed (x,y,z) system by a negative rotation about z' through the angle λ' (which brings x' into x), followed by a negative rotation about x through ω, or

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \mathcal{R}_1(-\omega)\mathcal{R}_3(-\lambda') \begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} x'\cos\lambda' + y'\sin\lambda' \\ x'\cos\omega\sin\lambda' + y'\cos\omega\cos\lambda' - z'\sin\omega \\ x'\sin\omega\sin\lambda' + y'\sin\omega\cos\lambda' + z'\cos\omega \end{bmatrix}. \quad (75)$$

Since the x' components of both forces contain $\cos\lambda'$ in their denominators, the x components will be independent of λ' while the y and z components will contain $\tan\lambda'$ as a factor. It is desireable to eliminate λ' altogether by substituting for $\tan\lambda'$ a function of the other angular variables. From the figure, using Napier's Analogies, we obtain

$$\tan\lambda' = \tan\vartheta\sin\omega.$$

Substituting this expression after the transformation (75) has been applied to equations (72) and (73) we have

$$\begin{bmatrix} F_{Ix} \\ F_{Iy} \\ F_{Iz} \end{bmatrix} = -\frac{I}{c}\pi a^2 P_{I'} \begin{bmatrix} \cos\vartheta \\ 0 \\ \sin\vartheta \end{bmatrix} \quad (76)$$

and

$$\begin{bmatrix} F_{Rx} \\ F_{Ry} \\ F_{Rz} \end{bmatrix} = -R_S\frac{I}{c}\pi a^2 \begin{bmatrix} P_{x'}\cos\vartheta \\ (P_{x'} - P_{z'})\sin\vartheta\sin\omega\cos\omega \\ P_{x'}\sin\vartheta\sin^2\omega + P_{z'}\sin\vartheta\cos^2\omega \end{bmatrix}. \quad (77)$$

Having expressed the forces in the (x,y,z) system, we have now arrived at essentially the same configuration as at the beginning of the previous subsection. Now, however, the x-axis is the spin axis. From figure 9 it can be seen that a negative rotation about y through ϑ will bring x into X', but this must be followed by a rotation about X' through π to complete the transformation to the (X',Y',Z') system. The second rotation merely changes the signs of the second and third components of the vector, so the complete transformation will be given by (63), if θ is replaced by ϑ and the signs of the second and third components of the final vector are changed. Applying such a transformation to (76) and (77) we have

$$\begin{bmatrix} F_{IX'} \\ F_{IY'} \\ F_{IZ'} \end{bmatrix} = -\frac{I}{c}\pi a^2 P_{I'} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

and

$$\begin{bmatrix} F_{RX'} \\ F_{RY'} \\ F_{RZ'} \end{bmatrix} = -R_S\frac{I}{c}\pi a^2 \begin{bmatrix} P_{x'} - (P_{x'} - P_{z'})\sin^2\vartheta\cos^2\omega \\ -(P_{x'} - P_{z'})\sin\vartheta\sin\omega\cos\omega \\ (P_{x'} - P_{z'})\sin\vartheta\cos\vartheta\cos^2\omega \end{bmatrix}$$

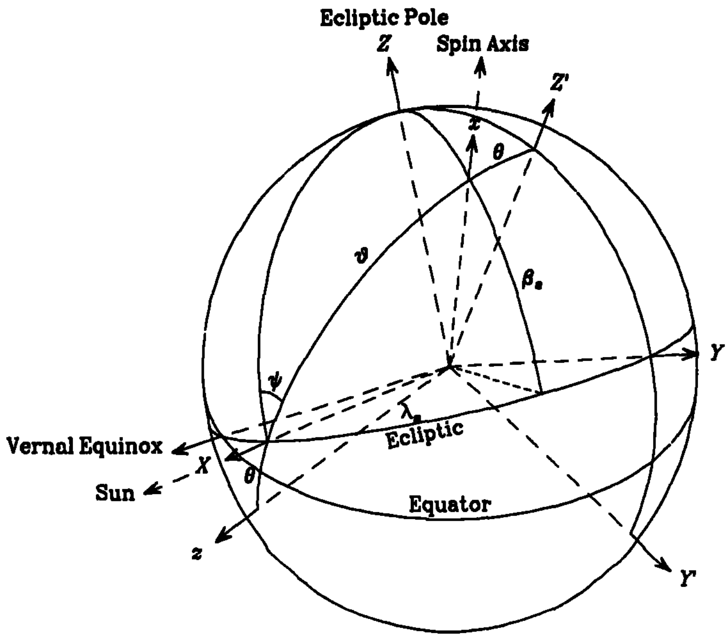

Figure 9.–Coordinate systems used in describing the radiation forces on a prolate spheroid rotating about a minor axis.

Using (74) it appears that the components of reflected force, in the (X', Y', Z') system, are proportional to

$$P_{X'} = P_{x'} - (P_{x'} - P_{z'}) \sin^2\vartheta \cos^2\omega$$

$$= \frac{e^2}{6}(1 - 3\sin^2\vartheta \cos^2\omega) - \frac{e^4}{48}(1 + 6\sin^2\vartheta\cos^2\omega - 15\sin^4\vartheta\cos^4\omega)$$

$$- \frac{e^6}{48}\left(1 - 3\sin^2\vartheta\cos^2\omega + 3\sin^4\vartheta\cos^4\omega - \frac{21}{5}\sin^6\vartheta\cos^6\omega\right) - \cdots,$$

$$P_{Y'} = -(P_{x'} - P_{z'}) \sin\vartheta \sin\omega \cos\omega$$

$$= -\frac{2e^2}{3}\sin\vartheta\sin\omega\cos\omega + \frac{e^4}{12}(\sin\vartheta\sin\omega\cos\omega + 3\sin^2\vartheta\sin\omega\cos^2\omega)$$

$$+ \frac{e^6}{12}\left(\sin\vartheta\sin\omega\cos\omega + \frac{3}{5}\sin^5\vartheta\sin\omega\cos^5\omega\right) + \cdots,$$

and

$$P_{Z'} = (P_{x'} - P_{z'}) \sin\vartheta\cos\vartheta\cos^2\omega$$

$$= \sin\vartheta\cos\vartheta\left(\frac{2e^2}{3}\cos^2\omega - \frac{e^4}{12}(\cos^2\omega + 3\sin^2\vartheta\cos^4\omega) - \frac{e^6}{12}\left(\cos^2\omega + \frac{3}{5}\sin^4\vartheta\cos^6\omega\right) - \cdots\right).$$

But ω, whose functions appear so frequently in these equations, has so far been treated as a fixed, but unknown, angle. In actuality ω is constantly changing as the spheroid rotates about its spin axis. If the spheroid is rotating slowly about its spin axis at a constant rate, it might be possible to determine the spin rate and phase angle by photometry, and then express the forces as functions of time. In all likelihood, however, the rotation period will be much smaller than the orbital period. If this is so, we can be satisfied with the mean value of the forces, the mean being computed over one rotation of the spheroid.

Using an overbar to represent the mean value over one rotation of a function of ω, we have for the term in the incident force

$$\begin{aligned}\bar{P}_I &= \frac{1}{2\pi}\int_0^{2\pi} P_{I'}\,d\omega \\ &= 1 - \frac{e^2}{2}(1 + \frac{1}{2}\sin^2\vartheta) - \frac{e^4}{8}(1 - \sin^2\vartheta + \frac{3}{8}\sin^4\vartheta) \\ &\quad - \frac{e^6}{16}(1 - \frac{1}{2}\sin^2\vartheta - \frac{3}{8}\sin^4\vartheta + \frac{5}{16}\sin^6\vartheta) - \cdots, \end{aligned} \tag{78}$$

and for the terms in the reflected force equation

$$\begin{aligned}\bar{P}_X &= \frac{e^2}{6}(1 - \frac{3}{2}\sin^2\vartheta) - \frac{e^4}{48}(1 - 3\sin^2\vartheta - \frac{45}{8}\sin^4\vartheta) \\ &\quad - \frac{e^6}{48}(1 - \frac{3}{2}\sin^2\vartheta + \frac{9}{8}\sin^4\vartheta - \frac{21}{16}\sin^6\vartheta) - \cdots, \end{aligned} \tag{79}$$

$$\bar{P}_Y = 0$$

and

$$\begin{aligned}\bar{P}_{Z'} = 2\sin\vartheta\cos\vartheta\Big(\frac{e^2}{6} &- \frac{e^4}{48}(1 + \frac{9}{4}\sin^2\vartheta) \\ &- \frac{e^6}{48}(1 + \frac{3}{8}\sin^4\vartheta) - \cdots\Big). \end{aligned} \tag{80}$$

The final transformation (64) is now used to express the forces in the desired coordinate system, (X,Y,Z). The incident force is again

$$F_I = -\frac{I}{c}\pi a^2 \bar{P}_I \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \tag{81}$$

and the reflected force is

$$F_R = -R_S\frac{I}{c}\pi a^2 \begin{bmatrix} \bar{P}_X \\ \bar{P}_{Z'}\sin\psi \\ \bar{P}_{Z'}\cos\psi \end{bmatrix}, \tag{82}$$

where $\bar{P}_I$, $\bar{P}_X$, and $\bar{P}_Z$ are given by (78) through (80) and the angles ϑ and ψ are obtained from (65).

With the exception of equation (69), series representations of the force components have been employed throughout this section. It should be pointed out that the exact expressions could have been used exclusively in treating rotation about the major axis. This was not done because the mathematics involved would be more tedious and the resulting equations would be far less appealing. But, assuming the existance of a spheroidal satellite which is rotating about its major axis, if the various parameters were known with sufficient precision to warrant the extra effort, and if its eccentricity were so large as to cause slow convergence of the series, then it is possible to derive exact expressions for the radiation forces. On the other hand, if exact expressions had been used in treating the case of rotation about a minor axis, elliptic integrals would have been encountered when we attempted to average the forces over a rotation of the spheroid, and we would then have been forced to resort to series. Closed form expressions are not possible for this case.

If the situation is further complicated by the spin axis precessing about the satellite–sun vector, as is suspected in the case of PAGEOS, no major modification of the equations developed in this section is required. The positions of the spin axis would have to be monitored over a period of time, and from these data and equations (65) the rate of the precession angle ω could be determined. Then, instead of using ω as a fixed angle in the force equations, it would be treated as a linear function of time.

Precession of the spin axis about any axis other than the satellite–sun vector would further complicate the force equations, but would not present any insurmountable difficulty. This circumstance is treated by Smith and Fea (1970). On the other hand, nutation of the spin axis would introduce periodic variations in both ϑ and the precession rate, which could not be described mathematically without further knowledge of the force, or forces, from which they arise. A treatment of spin axis nutation is beyond the scope of this report.

REFERENCES

Bryant, Robert W., "The Effect of Solar Radiation Pressure on the Motion of an Artificial Satellite," *NASA Technical Note D*–1063, National Aeronautics and Space Administration, Washington, D.C., September 1961, 8 pp.

Courant, R., *Differential and Integral Calculus, Volume II*, Interscience Publishers, Inc., New York, 1937, p. 274.

Fea, Kenneth H., and Smith, David E., "Some Further Studies of Perturbations of Satellites at Great Altitude:" *Planetary and Space Science*, Vol. 18, Pergamon Press, New York, 1970, pp. 1499–1525.

Georgevic, R.M. (Jet Propulsion Laboratory, California Institute of Technology, Pasadena, California), "The Solar Radiation Pressure Force and Torques Model," paper presented at AAS/AIAA Astrodynamics Specialists Conference, Fort Lauderdale, Florida, August 17–19, 1971, 21 pp.

Gradshteyn, I.S., and Ryzhik, I.M., *Table of Integrals, Series, and Products*, Fourth edition, Academic Press, New York, 1965, 1080 pp. (See p. 528). (Translation of *Tablitsy Integralov, Summ, Ryadov I Proievedeniy*, Gosudarstvennoe Izdatel'stvo, Fiziko-Matematicheskoy Literatury, Moscow, 1963).

Kozai, Yoshihide, "Effects of Solar Radiation Pressure on the Motion of an Artificial Satellite," *Smithsonian Contributions to Astrophysics*, Vol. 6, Smithsonian Institution, Astrophysical Observatory, Cambridge, Mass., 1963, pp. 109-112.

Lucas, James R., and Chovitz, Bernard H. (National Oceanic and Atmospheric Administration, Rockville, Maryland), "Tests on the Smith model for Perturbations of Balloon Satellites," paper presented at the Fifteenth COSPAR Meeting, Madrid, Spain, May 10-24, 1972, 13 pp.

Musen, Peter, "The Influence of Solar Radiation Pressure on the Motion of an Artificial Satellite," *Journal of Geophysical Research*, Vol. 65, No. 5, American Geophysical Union, Washington, D.C., May 1960, pp. 1391-1396.

Pierce, B.O., and Foster, Ronald M., *A Short Table of Integrals*, Fourth edition, Ginn and Company, New York, 1956, 189 pp.

Smith, David E., and Fea, Kenneth H., "Radiation Pressure Effects on the Acceleration of High Altitude Balloon Satellites," *GSFC Document No. X*-550-70-90, Goddard Space Flight Center, Greenbelt, Maryland, March 1970, 21 pp.

Smith, David E., and Kissell, K.E., "Anomalous Accelerations of the PAGEOS Spacecraft," *GSFC Document No. X*-553-71-338, Goddard Space Flight Center, Greenbelt, Maryland, August 1971, 6 pp.

Thekaekara, M.P., and Drummond, A.J., "Standard Values for the Solar Constant and its Spectral Components," *Nature Physical Science*, Vol. 229, January 4, 1971, pp. 6-9.

Vanderburgh, R.C., and Kissell, K.E. (Aerospace Research Laboratories, USAF, Wright-Patterson AFB, Ohio), "Measurements of Deformation and Spin Dynamics of the PAGEOS Balloon-Satellite by Photoelectric Photometry," paper presented at the Eleventh Astrodynamics Conference, Goddard Space Flight Center, Greenbelt, Maryland, April 1970, 15 pp.

APPENDIX I. EVALUATION OF INTEGRALS ASSOCIATED WITH INCIDENT FORCE

Evaluation of $S_2[z \cot\alpha \sin\gamma]$

Equations (23) and (30) provide $\cot\alpha$ and $\sin\gamma$, respectively, as functions of the integration variable z. Substituting these expressions into the argument, the operator becomes

$$S_2[z \cot\alpha \sin\gamma] = \frac{a(1-e^2\sin^2\theta)^{1/2}}{b\cos\theta}\int_{-\eta}^{+\eta}(\eta^2-z^2)^{1/2}\,dz. \tag{83}$$

This integrand can be found in any book of integral tables. For example Pierce and Foster (1956), formula 127 gives

$$\int(\eta^2-z^2)^{1/2}\,dz = \frac{z}{2}(\eta^2-z^2)^{1/2} + \frac{\eta^2}{2}\sin^{-1}\frac{z}{\eta},$$

or

$$\int_{-\eta}^{+\eta}(\eta^2-z^2)^{1/2}\,dz = \frac{\pi\eta^2}{2} = \frac{\pi a^2\cos^2\theta}{2(1-e^2\sin^2\theta)}. \tag{84}$$

Substituting into (83) we have

$$S_2[z \cot\alpha \sin\gamma] = \frac{\pi a^3\cos\theta}{2b(1-e^2\sin^2\theta)^{1/2}}. \tag{85}$$

Evaluation of $S_1[z]$

From the definition, equation (31), the operator represents the sum of double integrals

$$\begin{aligned} S_1[z] &= \int_{-\pi}^{+\pi}\int_{\eta}^{a} z\,dz\,d\lambda + \int_{-\gamma}^{+\gamma}\int_{-\eta}^{+\eta} z\,dz\,d\lambda \\ &= 2\pi\int_{\eta}^{a} z\,dz + 2\int_{-\eta}^{+\eta}\gamma z\,dz. \end{aligned} \tag{86}$$

The first integral in this expression is straightforward, but the second must be handled by parts, i.e.,

$$2\int\gamma z\,dz = \gamma z^2 - \int z^2\,d\gamma.$$

For the integration limits, a check of equations (29) and (30) shows that when $z=\eta$, $\gamma=\pi$, and when $z=-\eta$, $\gamma=0$. Therefore,

$$2\int_{-\eta}^{+\eta}\gamma z\,dz = \pi\eta^2 - \int_0^{\pi} z^2\,d\gamma,$$

or, substituting into (86) we have

$$S_1[z] = \pi a^2 - \int_0^{\pi} z^2\, d\gamma. \tag{87}$$

It is possible to express z^2 as a function of γ and evaluate the remaining integral directly, but it is simpler to perform a change of integration variable. Using the standard formula for differentiating the arccosine function,

$$\frac{d\gamma}{dz} = -\csc\gamma \frac{d}{dz}\cos\gamma = \frac{ab\tan\theta}{(a^2-z^2)^{3/2}\sin\gamma},$$

where the term on the right is obtained by differentiating (29). Now using (30) we have

$$d\gamma = \frac{ab\sin\theta}{(1-e^2\sin^2\theta)^{1/2}(a^2-z^2)(\eta^2-z^2)^{1/2}}\, dz. \tag{88}$$

Therefore,

$$\int_0^{\pi} z^2\, d\gamma = \frac{ab\sin\theta}{(1-e^2\sin^2\theta)^{1/2}} \int_{-\eta}^{+\eta} \frac{z^2\, dz}{(a^2-z^2)(\eta^2-z^2)^{1/2}}, \tag{89}$$

which does not seem to simplify the task appreciably. However, this integral can be written as the difference of two simpler forms using

$$\frac{z^2\, dz}{(a^2-z^2)(\eta^2-z^2)^{1/2}} = \frac{a^2\, dz}{(a^2-z^2)(\eta^2-z^2)^{1/2}} - \frac{dz}{(\eta^2-z^2)^{1/2}}.$$

Both of these forms can be found in integral tables, but if we solve (88) for dz as a function of $d\gamma$, and substitute into the first term we have

$$\int_0^{\pi} z^2\, dz = a^2 \int_0^{\pi} d\gamma - \frac{ab\sin\theta}{(1-e^2\sin^2\theta)^{1/2}} \int_{-\eta}^{+\eta} \frac{dz}{(\eta^2-z^2)^{1/2}}.$$

Pierce and Foster (1956), formula 130, gives

$$\int \frac{dz}{(\eta^2-z^2)^{1/2}} = \sin^{-1}\frac{z}{\eta} \qquad \text{or} \qquad \int_{-\eta}^{+\eta} \frac{dz}{(\eta^2-z^2)^{1/2}} = \pi. \tag{90}$$

Hence

$$\int_0^{\pi} z^2\, d\gamma = \pi a^2 - \frac{\pi ab\sin\theta}{(1-e^2\sin^2\theta)^{1/2}}, \tag{91}$$

which substituted into (87) yields

$$S_1[z] = \frac{\pi ab\sin\theta}{(1-e^2\sin^2\theta)^{1/2}}. \tag{92}$$

APPENDIX II. EVALUATION OF INTEGRALS ASSOCIATED WITH REFLECTED FORCE

Evaluation of $S_2[z \sin\alpha \cos\alpha \sin\gamma]$

Substituting equations (24) and (30) into the argument of this operator, and using the definition (32) we have

$$S_2[z \sin\alpha \cos\alpha \sin\gamma] = \frac{UV}{\cos\theta} \int_{-\eta}^{+\eta} \frac{z(\eta^2-z^2)^{1/2}}{a^2-e^2z^2}\, dz, \tag{93}$$

where

$$U = (1-e^2)^{1/2} \quad \text{and} \quad V = (1-e^2\sin^2\theta)^{1/2}.$$

This integrand can be put into more tractable form by eliminating both z^2 and the radical from the numerator. First, we can use

$$\frac{z^2(\eta^2-z^2)^{1/2}}{a^2-e^2z^2} = -\frac{1}{e^2}\left[(\eta^2-z^2)^{1/2} - \frac{a^2(\eta^2-z^2)^{1/2}}{a^2-e^2z^2}\right]$$

$$= -\frac{1}{e^2}\left[(\eta^2-z^2)^{1/2} - \frac{a^2\eta^2}{(a^2-e^2z^2)(\eta^2-z^2)^{1/2}} + \frac{a^2z^2}{(a^2-e^2z^2)(\eta^2-z^2)^{1/2}}\right].$$

But the second step, which removed the radical from the numerator, produced a term with z^2 in its numerator. We can now use

$$\frac{a^2z^2}{(a^2-e^2z^2)(\eta^2-z^2)^{1/2}} = \frac{a^4}{e^2(a^2-e^2z^2)(\eta^2-z^2)^{1/2}} - \frac{a^2}{e^2(\eta^2-z^2)^{1/2}},$$

which substituted into the above expression produces

$$\int_{-\eta}^{+\eta} \frac{z^2(\eta^2-z^2)^{1/2}}{a^2-e^2z^2}\, dz = \frac{1}{e^2}\int_{-\eta}^{+\eta}(\eta^2-z^2)^{1/2}dz + \frac{a^2}{e^4}\int_{-\eta}^{+\eta}\frac{dz}{(\eta^2-z^2)^{1/2}}$$

$$- \frac{a^2(a^2-e^2\eta^2)}{e^4}\int_{-\eta}^{+\eta}\frac{dz}{(a^2-e^2z^2)(\eta^2-z^2)^{1/2}}. \tag{94}$$

The first two of these integrals were evaluated in Appendix I, equations (84) and (90), and for the third we can use formula 239 of Pierce and Foster (1956) which is

$$\int \frac{dz}{(a^2-e^2z^2)(\eta^2-z^2)^{1/2}} = \frac{1}{a(a^2-e^2\eta^2)^{1/2}} \tan^{-1}\frac{z(a^2-e^2\eta^2)^{1/2}}{a(\eta^2-z^2)^{1/2}}.$$

When the limits are employed, it is found that the denominator in the arctangent argument is zero for both limits, but the numerator is positive when $z = \eta$ and negative when $z = -\eta$. By integrating over the range from $-\eta + \varepsilon$ to $\eta - \varepsilon$ and taking the limit as ε approaches zero we can be assured that

$$\int_{-\eta}^{+\eta} \frac{dz}{(a^2-e^2z^2)(\eta^2-z^2)^{1/2}} = \frac{\pi}{a(a^2-e^2\eta^2)^{1/2}} = \frac{\pi V}{a^2 U}. \tag{95}$$

Substituting (84), (90), and (95) into (94) obtains

$$\int_{-\eta}^{+\eta} \frac{z^2(\eta^2-z^2)^{1/2}}{a^2-e^2z^2}\, dz = -\frac{\pi a^2 \cos^2\theta}{2e^2V^2} + \frac{\pi a^2}{e^4} - \frac{\pi a^2 U}{e^4 V},$$

and from (93)

$$S_2[z \sin\alpha \cos\alpha \sin\gamma] = \frac{\pi a^2 U \cos\theta}{2e^2 V} - \frac{\pi a^2(U^2 - UV)}{e^4 \cos\theta}. \tag{96}$$

Evaluation of $S_1[z \sin^2\alpha]$

Again we have a sum of double integrals

$$\begin{aligned} S_1[z \sin^2\alpha] &= \int_{-\pi}^{+\pi}\int_{\eta}^{a} z \sin^2\alpha\, dz\, d\lambda + \int_{-\gamma}^{+\gamma}\int_{-\eta}^{+\eta} z \sin^2\alpha\, dz\, d\lambda \\ &= 2\pi \int_{\eta}^{a} z \sin^2\alpha\, dz + 2\int_{-\eta}^{+\eta} \gamma z \sin^2\alpha\, dz. \end{aligned} \tag{97}$$

The second term can be integrated by parts and in the process provides some simplification. Let q represent the indefinite integral

$$q = \int z \sin^2\alpha\, dz,$$

and we can write

$$\int \gamma z \sin^2\alpha\, dz = \gamma q - \int q\, d\gamma. \tag{98}$$

As was pointed out in Appendix I, when $z = \eta$, $\gamma = \pi$, and when $z = -\eta$, $\gamma = 0$. Since q is the indefinite form of the first integral in (97), using (98) and employing these limits obtains

$$S_1[z \sin^2\alpha] = 2\pi\, q(a) - 2\int_0^{\pi} q\, d\gamma. \tag{99}$$

Now, to evaluate q we can substitute (24) to obtain the integrand as a function of z, and then use a change of variable. Let $\rho = z^2$. Then $d\rho = 2z\, dz$, and we have

$$q = U^2 \int \frac{z^3\, dz}{a^2-e^2z^2} = \frac{U^2}{2}\int \frac{\rho\, d\rho}{a^2-e^2\rho}.$$

Formula 31 of Pierce and Foster (1956) gives

$$\int \frac{\rho\, d\rho}{a^2-e^2\rho} = \frac{1}{e^4}\,[a^2 - e^2\rho - a^2\ln(a^2-e^2\rho)].$$

Since $\rho = z^2$ we have

$$q = \frac{U^2}{2e^4}\,[a^2 - e^2z^2 - a^2\ln(a^2-e^2z^2)],$$

which substituted into (99) yields

$$S_1[z\sin^2\alpha] = \frac{\pi a^2U^2}{e^4} - \frac{\pi a^2U^2}{e^2} - \frac{\pi a^2U^2}{e^4}\ln a^2U^2 - \frac{a^2U^2}{e^4}\int_0^\pi d\gamma$$
$$+ \frac{U^2}{e^2}\int_0^\pi z^2 d\gamma + \frac{a^2U^2}{e^4}\int_0^\pi \ln(a^2-e^2z^2)\,d\gamma.$$

Integration of the fourth term shows that it cancels the first term, and the fifth term is an integral that has already been evaluated in (91). Hence

$$S_1[z\sin^2\alpha] = -\frac{\pi a^2U^3\sin\theta}{e^2V} - \frac{\pi a^2U^2}{e^4}\ln(a^2U^2) + \frac{a^2U^2}{e^4}\int_0^\pi \ln(a^2-e^2z^2)\,d\gamma, \quad (100)$$

and we are left with a single integral which, unfortunately, is not in a very convenient form. The best approach appears to be to express the integrand as a function of γ. To this end we square equation (29) and solve for

$$z^2 = \frac{a^2\cos^2\theta\cos^2\gamma}{U^2\sin^2\theta + \cos^2\theta\cos^2\gamma},$$

which is employed to obtain

$$a^2-e^2z^2 = \frac{a^2U^2\sin^2\theta + a^2\cos^2\theta\cos^2\gamma - a^2e^2\cos^2\theta\cos^2\gamma}{U^2\sin^2\theta + \cos^2\theta\cos^2\gamma}$$
$$= \frac{a^2U^2(\sin^2\theta + \cos^2\theta\cos^2\gamma)}{U^2\sin^2\theta\,(1 + \dfrac{\cos^2\theta}{U^2\sin^2\theta}\cos^2\gamma)}$$
$$= \frac{a^2(1 - \cos^2\theta\sin^2\gamma)}{\sin^2\theta\,(1 + \dfrac{\cos^2\theta}{U^2\sin^2\theta}\cos^2\gamma)}.$$

While this is not a very appealing expression, its logarithm can be written as the sum of four logarithms of which two are independent of γ and the other two are of similar form. Hence

$$\int_0^\pi \ln(a^2-e^2z^2)\,d\gamma = \pi\ln a^2 - \pi\ln\sin^2\theta + \int_0^\pi \ln(1-\cos^2\theta\sin^2\gamma)\,d\gamma$$
$$- \int_0^\pi \ln(1 + \frac{\cos^2\theta}{U^2\sin^2\theta}\cos^2\gamma)\,d\gamma. \quad (101)$$

Neither of these two forms can be found in tables of indefinite integrals, but in the definite integral section of Gradshteyn and Ryzhik (1965) formula 4.226.2 states that

$$\int_0^\pi \ln(1+A\sin^2 x)\,dx = \int_0^\pi \ln(1+A\cos^2 x)\,dx = 2\pi\ln\frac{1+(1+A)^{1/2}}{2},$$

where $A \geqq -1$. Obviously $-\cos^2 Q \geqq -1$ and since $\theta \leqq \pi/2$, $U^{-2}\cot^2\theta \geqq -1$. Hence

$$\int_0^\pi \ln(1-\cos^2\theta\sin^2\gamma)\,d\gamma = 2\pi\ln\frac{1+\sin\theta}{2},$$

and

$$\int_0^\pi \ln\left(1+\frac{\cos^2\theta}{U^2\sin^2\theta}\cos^2\gamma\right)d\gamma = 2\pi\ln\frac{U\sin\theta+(U^2\sin^2\theta+\cos^2\theta)^{1/2}}{2U\sin\theta}$$

$$= 2\pi\ln\frac{U\sin\theta+V}{2U\sin\theta}.$$

Substituting these expressions into (101) obtains

$$\int_0^\pi \ln(a^2-e^2z^2)\,d\gamma = \pi\ln a^2 - \pi\ln\sin^2\theta + 2\pi\ln(1+\sin\theta)$$

$$-\,2\pi\ln(V+U\sin\theta) + 2\pi\ln U + 2\pi\ln\sin\theta$$

$$= \pi\ln a^2U^2 - 2\pi\ln\left(\frac{V+U\sin\theta}{1+\sin\theta}\right),$$

and from (100)

$$S_\iota[z\sin^2\alpha] = -\frac{\pi a^2U^3\sin\theta}{e^2V} - \frac{2\pi a^2U^2}{e^4}\ln\left(\frac{V+U\sin\theta}{1+\sin\theta}\right). \tag{102}$$

☆ U. S. GOVERNMENT PRINTING OFFICE : 1974 O - 551-337

(Continued from inside front cover)

NOAA TECHNICAL REPORTS

NOS 41 A User's Guide to a Computer Program for Harmonic Analysis of Data at Tidal Frequencies. R. E. Dennis and E. E. Long, July 1971. Price $0.65 (COM-71-50606)

NOS 42 Computational Procedures for the Determination of a Simple Layer Model of the Geopotential From Doppler Observations. Bertold U. Witte, April 1971. Price $0.65 (COM-71-50400)

NOS 43 Phase Correction for Sun-Reflecting Spherical Satellite. Erwin Schmid, August 1971. Price $0.25 (COM-72-50080)

NOS 44 The Determination of Focal Mechanisms Using P- and S-Wave Data. William H. Dillinger, Allen J. Pope, and Samuel T. Harding, July 1971. Price $0.60 (COM-71-50392)

NOS 45 Pacific SEAMAP 1961-70 Data for Area 15524-10: Longitude 155°W to 165°W, Latitude 24°N to 30°N, Bathymetry, Magnetics, and Gravity. J. J. Dowling, E. E. Chiburis, P. Dehlinger, and M. J. Yellin, January 1972. Price $3.50 (COM-72-51029)

NOS 46 Pacific SEAMAP 1961-70 Data for Area 15530-10: Longitude 155°W to 165°W, Latitude 30°N to 36°N, Bathymetry, Magnetics, and Gravity. J. J. Dowling, E. F. Chiburis, P. Dehlinger, and M. J. Yellin, January 1972. Price $3.50

NOS 47 Pacific SEAMAP 1961-70 Data for Area 15248-14: Longitude 152°W to 166°W, Latitude 48°N to 54°N, Bathymetry, Magnetics, and Gravity. J. J. Dowling, E. F. Chiburis, P. Dehlinger, and M. J. Yellin, April 1972. Price $3.50 (COM-72-51030)

NOS 48 Pacific SEAMAP 1961-70 Data for Area 16648-14: Longitude 166°W to 180°, Latitude 48°N to 54°N, Bathymetry, Magnetics, and Gravity. J. J. Dowling, E. F. Chiburis, P. Dehlinger, and M. J. Yellin, April 1972. Price $3.00 (COM-72-51028)

NOS 49 Pacific SEAMAP 1961-70 Data for Areas 16530-10 and 17530-10: Longitude 165°W to 180°, Latitude 30°N to 36°N, Bathymetry, Magnetics, and Gravity. E. F. Chiburis, J. J. Dowling, P. Dehlinger, and M. J. Yellin, July 1972. Price $4.75

NOS 50 Pacific SEAMAP 1961-70 Data for Areas 16524-10 and 17524-10: Longitude 165°W to 180°, Latitude 24°N to 30°N, Bathymetry, Magnetics, and Gravity. E. F. Chiburis, J. J. Dowling, P. Dehlinger, and M. J. Yellin, July 1972. Price $5.75

NOS 51 Pacific SEAMAP 1961-70 Data for Areas 15636-12, 15642-12, 16836-12, and 16842-12: Longitude 156°W to 180°, Latitude 36°N to 48°N, Bathymetry, Magnetics, and Gravity. E. F. Chiburis, J. J. Dowling, P. Dehlinger, and M. J. Yellin, July 1972. Price $11.00 (COM-73-50280)

NOS 52 Pacific SEAMAP 1961-70 Data Evaluation Summary. P. Dehlinger, E. F. Chiburis, and J. J. Dowling, July 1972. Price $0.40

NOS 53 Grid Calibration by Coordinate Transfer. Lawrence W. Fritz, November 1972. (COM-73-50240)

NOS 54 A Cross-Coupling Computer for the Oceanographer's Askania Gravity Meter. Carl A. Pearson and Thomas E. Brown, November 1972. (COM-73-50317)

NOS 55 A Mathematical Model for the Simulation of a Photogrammetric Camera Using Stellar Control. Chester C Slama, December 1972.

NOS 56 Cholesky Factorization and Matrix Inversion. Erwin Schmid, March 1973. (COM-73-50486)

NOS 57 Complete Comparator Calibration. Lawrence W. Fritz, July 1973. Price $1.30.

NOS 58 Telemetering Hydrographic Tide Gauge. Charles W. Iseley, July 1973. Price $0.50 (COM-74-50001)

NOS 59 Gravity Gradients at Satellite Altitudes. B. Chovitz, J. Lucas, and F. Morrison, November 1973. Price $0.55.

NOS 60 The Reduction of Photographic Plate Measurements for Satellite Triangulation. Anna-Mary Bush, June 1973. Price $2.10 (COM-73-50749)